CHAPTER 7

Linear Programming

7.1. INTRODUCTION

Linear programming (briefly written as LP) came into existence during World War II (1939-45) when the British and American military management called upon a group of scientists to study and plan the war activities so that maximum damages could be inflicted on the enemy camps at minimum cost and loss with the limited resources available with them. Because of the success in military operations, the subject was found extremely useful for allocation of scarce resources for optimal results. Business and industry, agriculture and military sectors have, however, made the most significant use of the technique. But now it is being extensively used in all functional areas of management, airlines, oil refining, education, pollution control, transportation planning, health care system etc. The utility of the technique is enhanced by the availability of highly efficient computer codes. A lot of research work is being carried out all over the world. Kantorovich and Koopmans were awarded the noble prize in the year 1975 in economics for their pioneering work in linear programming. In India, it came into existence in 1949, with the opening of an operations research unit at the Regional Research Laboratory at Hyderabad.

7.2. DEFINITIONS AND BASIC CONCEPTS

The word '*programming*' means planning and refers to a process of determining a particular program or strategy or course of action among various alternatives to achieve the desired objective. The word **'linear'** means that all relationships involved in a particular programme are linear.

Linear Programming is the technique of optimizing (*i.e.*, maximizing or minimizing) a linear function of several variables subject to a number of constraints stated in the form of linear inequations/equations.

Linear Programming Problem. Any problem in which we apply linear programming is called a linear programming problem (briefly written as LPP).

The mathematical model of a general linear programming problem with n variables and m constraints can be stated as

Optimize (maximize or minimize)

$$Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$$

subject to

$$\left.\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n\ (\le, =, \ge)\ b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n\ (\le, =, \ge)\ b_2 \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n\ (\le, =, \ge)\ b_m \end{aligned}\right\} \quad \ldots(1)$$

and $\quad x_1, x_2, \ldots\ldots, x_n \ge 0 \quad \ldots(2)$

where

(*i*) the linear function Z which is to be maximized or minimized is called the **objective function** of the LPP.

(*ii*) $x_1, x_2, \ldots\ldots, x_n$ are called **decision** or **structural variables.**

(*iii*) The equations/inequations (1) are called the **constraints** of LPP. Due to limited resources, there is always a stage beyond which we cannot pursue our objective. Any linear equation or inequation in one or more decision variables is called a linear constraint.

(*iv*) The expression $(\le, =, \ge)$ means that each constraint may take any one of the three signs.

(*v*) The set of inequations (2) is known as the set of **non-negative restrictions** of the general LPP.

(*vi*) The constant c_j ($j = 1, 2, \ldots\ldots, n$) represents the contribution to the objective function of the *j*th variable.

(*vii*) b_i ($i = 1, 2, \ldots\ldots, m$) is the constant representing the requirement or availability of the *i*th constraint.

(*viii*) a_{ij} = ($i = 1, 2, \ldots\ldots, m; j = 1, 2, \ldots\ldots, n$) is referred to as the technological co-efficient.

Now, we give below some definitions and concepts which pertain to general LPP.

Solution. A set of values of the decision variables which satisfy all the constraints of an LPP is called a **solution** to that problem.

Feasible Solution. Any solution of an LPP which also satisfies the non-negativity restrictions of the problem is called a **feasible solution** to that problem.

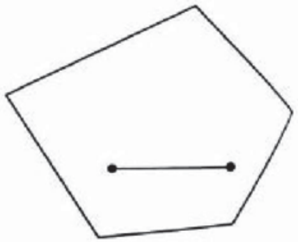
Feasible Region. The set of all feasible solutions of an LPP is called its **feasible region.**

Optimal (or Optimum) Solution. Any feasible solution which optimizes the objective function of an LPP is called **an optimal (or optimum) solution** of the LPP. We have used the word 'an' with optimal solution. The reason being that an LPP may have many optimal solutions.

The value of the objective function Z at an optimal solution is called an **optimal value.**

Optimization Technique. The process of obtaining an optimal value is called **optimization technique.**

Convex Region. A region is said to be convex if the line segment joining any two arbitrary points of the region lies entirely within the region. The feasible region of an LPP is always a polyhedral convex region, *i.e.*, a convex region whose boundary consists of line segments.



7.3. FORMULATION OF A LINEAR PROGRAMMING PROBLEM

Formulation of an LPP as a mathematical model is the first and the most important step in the solution of the LPP. It is an art in itself and needs sufficient practice. However, in general, the following steps are involved:

1. Identify the decision variables and denote them by $x_1, x_2, x_3, \ldots\ldots$.
2. Identify the objective function and express it as a linear function of the decision variables. In its general form, it is represented as:

   Optimize (Maximize or Minimize) $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n$

   (The optimal value of Z is obtained by the graphical method or simplex method. The graphical method is more suitable when there are two variables.)
3. Identify the set of constraints and express them as linear equations/inequations in terms of decision variables.
4. Add the non-negativity restrictions on the decision variables, as in the physical problems, negative values of decision variables have no valid interpretation.

   Thus $x_1 \ge 0, \quad x_2 \ge 0, \ldots\ldots, x_n \ge 0.$

ILLUSTRATIVE EXAMPLES

Example 1. *A dealer wishes to purchase a number of fans and sewing machines. He has only ₹9750 to invest and has space for at most 30 items. A fan costs him ₹480 and a sewing machine ₹360. His expectation is that he can sell a fan at a profit of ₹35 and a sewing machine at a profit of ₹24. Assume that he can sell all the items that he buys. Formulate this problem as an LPP, so that he can maximize his profit.*

Sol. Let x and y denote the number of fans and sewing machines respectively. (x and y are the decision variables)

Cost of x fans = ₹ $480x$

Cost of y sewing machines = ₹ $360y$

$\Rightarrow$ The total cost of x fans and y sewing machines = ₹ $(480x + 360y)$

Since the dealer has only ₹ 9750 to invest, the total cost cannot exceed ₹ 9750.

$\therefore \quad 480x + 360y \le 9750$

Dividing by 30, we have $\mathbf{16x + 12y \le 325}$

Since the dealer has space for at most 30 items

$\therefore \quad \mathbf{x + y \le 30}$

Since the number of fans and the number of sewing machines cannot be negative

$\therefore \quad \mathbf{x \ge 0, \quad y \ge 0} \quad$ (non-negativity constraints)

Profit on x fans = ₹ $35x$

Profit on y sewing machines = ₹ $24y$

$\Rightarrow$ The total profit on x fans and y sewing machines

= ₹ $(35x + 24y)$

The dealer wishes to maximize his profit

$Z = 35x + 24y$ (objective function)

$\therefore$ The mathematical formulation of the LPP is

Maximize $\quad Z = 35x + 24y$

subject to the constraints

$$16x + 12y \le 325$$
$$x + y \le 30$$
$$x \ge 0, \quad y \ge 0.$$

Example 2. *A manufacturer of patent medicines is preparing a production plan on medicines A and B. There are sufficient raw materials available to make 20000 bottles of A and 40000 bottles of B, but there are only 45000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of A, it takes 1 hour to prepare enough material to fill 1000 bottles of B and there are 66 hours available for this operation. The profit is ₹8 per bottle for A and ₹7 per bottle for B. The manufacturer wants to schedule his production in order to maximize his profit. Formulate this problem as an LP model.*

Sol. Let x and y denote the number of bottles of type A and type B medicines respectively.

Total profit (in ₹) is $Z = 8x + 7y$

Raw material constraints are

$$x \le 20000, \quad y \le 40000$$

Since only 45000 bottles are available

$\therefore$ $$x + y \le 45000$$

It takes 3 hours to prepare enough material to fill 1000 bottles of type A.

$\therefore$ The number of hours required to prepare enough material to fill x bottles of type A $= \frac{3x}{1000}$.

Similarly, the number of hours required to prepare enough material to fill y bottles of type B $= \frac{y}{1000}$.

Since total number of hours available for this operation is 66

$\therefore$ $$\frac{3x}{1000} + \frac{y}{1000} \le 66 \quad \text{or} \quad 3x + y \le 66000$$

Obviously $\quad x \ge 0, \; y \ge 0$

The LP model of the problem is

Maximize $\quad Z = 8x + 7y$

subject to the constraints

$$x \le 20000$$
$$y \le 40000$$
$$x + y \le 45000$$
$$3x + y \le 66000$$
$$x \ge 0, \; y \ge 0.$$

Example 3. *An electronic company produces three types of parts for automatic washing machines. It purchases casting of the parts from a local foundry and then finishes the part on drilling, shaping and polishing machines.*

The selling prices of parts A, B and C respectively are ₹8, ₹10 and ₹14. All parts made can be sold. Castings for parts A, B and C respectively cost ₹5, ₹6 and ₹10. The company possesses only one of each type of machine. Costs per hour to run each of the three machines are ₹20 for drilling, ₹30 for shaping and ₹30 for polishing. The capacities (parts per hour) for each part on each machine are shown in the following table:

Machine	*Capacity per hour*		
	Part A	*Part B*	*Part C*
Drilling	25	40	25
Shaping	25	20	20
Polishing	40	30	40

The management of the shop wants to know how many parts of each type it should produce per hour in order to maximize profit for an hour's run. Formulate this problem as an LP model so as to maximize total profit to the company.

Sol. Let x_1, x_2 and x_3 denote respectively the numbers of type A, B and C parts to be produced per hour.

Consider one type A part

Selling price $= ₹\,8$

Cost of casting $= ₹\,5$

Since 25 type A parts per hour can be run on the drilling machine at a cost of ₹ 20

$\therefore$ Drilling cost per type A part $= ₹\,\frac{20}{25} = ₹\,0.80$

Similarly shaping cost $= ₹\,\frac{30}{25} = ₹\,1.20$

Polishing cost $= ₹\,\frac{30}{40} = ₹\,0.75$

$\therefore$ **Profit per type A part** $= ₹\,[8 - (5 + 0.80 + 1.20 + 0.75)] = ₹\,0.25$

By similar reasoning

Profit per type B part $= ₹\left[10 - \left(6 + \frac{20}{40} + \frac{30}{20} + \frac{30}{30}\right)\right] = ₹\,1.00$

Profit per type C part $= ₹\left[14 - \left(10 + \frac{20}{25} + \frac{30}{20} + \frac{30}{40}\right)\right] = ₹\,0.95$

$\therefore$ Total profit is given by $Z = 0.25x_1 + 1.00x_2 + 0.95x_3$.

On the drilling machine, one type A part consumes $\frac{1}{25}$th of the available hour, one type B part consumes $\frac{1}{40}$th and one type C part consumes $\frac{1}{25}$th of the available hour.

$\therefore$ **The drilling machine constraint** is

$$\frac{x_1}{25} + \frac{x_2}{40} + \frac{x_3}{25} \le 1$$

Similarly, the shaping machine constraint is

$$\frac{x_1}{25} + \frac{x_2}{20} + \frac{x_3}{20} \le 1$$

and the polishing machine constraint is

$$\frac{x_1}{40}+\frac{x_2}{30}+\frac{x_3}{40}\le 1$$

Also the non-negativity constraint is

$$x_1, x_2, x_3 \ge 0$$

$\therefore$ The LP model of the given problem is:

Maximize $\quad Z = 0.25x_1 + 1.00x_2 + 0.95x_3$

subject to the constraints

$$\frac{x_1}{25}+\frac{x_2}{40}+\frac{x_3}{25}\le 1$$
$$\frac{x_1}{25}+\frac{x_2}{20}+\frac{x_3}{20}\le 1$$
$$\frac{x_1}{40}+\frac{x_2}{30}+\frac{x_3}{40}\le 1$$
$$x_1, x_2, x_3 \ge 0.$$

Example 4. *Consider the following problem faced by a production planner in a soft drink plant. He has two bottling machines A and B. A is designed for 8-ounce bottles and B for 16-ounce bottles. However, each can also be used for both types of bottles with some loss of efficiency. The manufacturing data is as follows:*

Machine	*8-ounce Bottles*	*16-ounce Bottles*
A	*100/minute*	*40/minute*
B	*60/minute*	*75/minute*

The machines can be run for 8 hours per day, 5 days per week. Profit on an 8-ounce bottle is 15 paise and on a 16-ounce bottle 25 paise. Weekly production of the drink cannot exceed 3,00,000 bottles and the market can absorb 25,000, 8-ounce bottles and 7,000, 16-ounce bottles per week. The planner wishes to maximize his profit, subject of course, to all the production and marketing restrictions. Formulate this problem as an LP model to maximize total profit.

Sol. Let x_1 and x_2 denote the number of 8-ounce and 16-ounce bottles respectively to be produced weekly.

Total profit (in ₹) is given by $\quad Z = 0.15x_1 + 0.25x_2$

Machine time constraints

Total available time in a week on **machine A** = 8 × 5 × 60 minutes = 2400 minutes

Time for 100, 8-ounce bottles is 1 minute

$\therefore$ Time for x_1 8-ounce bottles is $\frac{x_1}{100}$ minutes

Similarly, time for x_2 16-ounce bottles is $\frac{x_2}{40}$

$\Rightarrow$ Machine time constraint on machine A is

$$\frac{x_1}{100}+\frac{x_2}{40}\le 2400$$

Similarly, machine time constraint on machine B is

$$\frac{x_1}{60}+\frac{x_2}{75}\le 2400$$

Production constraint. Since weekly production cannot exceed 3,00,000

$$x_1 + x_2 \le 3,00,000$$

$\therefore$

Marketing constraints. The market can absorb 25,000, 8-ounce bottles and 7,000, 16-ounce bottles

$$x_1 \le 25,000 \quad\text{and}\quad x_2 \le 7,000.$$

$\therefore$

Non-negativity constraints. The number of bottles cannot be negative

$$x_1 \ge 0, \quad x_2 \ge 0$$

$\therefore$

Hence the LP model of the given problem is:

Maximize $\quad Z = 0.15x_1 + 0.25x_2$

subject to the constraints

$$\frac{x_1}{100}+\frac{x_2}{40}\le 2400$$
$$\frac{x_1}{60}+\frac{x_2}{75}\le 2400$$
$$x_1 + x_2 \le 3,00,000$$
$$x_1 \le 25000, \quad x_2 \le 7000$$
$$x_1 \ge 0, \quad x_2 \ge 0.$$

Example 5. *A firm making castings uses electric furnace to melt iron with the following specifications:*

	Minimum	*Maximum*
Carbon	*3.20%*	*3.40%*
Silicon	*2.25%*	*2.35%*

Specifications and costs of various raw materials used for this purpose are given below:

Material	*Carbon %*	*Silicon %*	*Cost (₹)*
Steel scrap	*0.4*	*0.15*	*850/tonne*
Cast iron scrap	*3.80*	*2.40*	*900/tonne*
Remelt from foundry	*3.50*	*2.30*	*500/tonne*

If the total charge of iron metal required is 4 tonnes, find the weight in kg of each raw material that must be used in the optimal mix at minimum cost. Formulate this problem as an LP model.

Sol. Let x_1, x_2, x_3 be the weights (in kg) of the raw materials:

Steel scrap, cast iron scrap and remelt from foundry respectively.

Cost of 1 tonne *i.e.*, 1000 kg steel scrap is ₹ 850

$\Rightarrow$ cost of x_1 kg steel scrap $= ₹\ \frac{850}{1000}x_1 = ₹\ 0.85x_1$

Similarly, cost of x_2 kg of cast iron scrap $= ₹\ \frac{900}{1000}x_2 = ₹\ 0.9x_2$

and cost of x_3 kg of remelt from foundry $= ₹\ \frac{500}{1000}x_3 = ₹\ 0.5x_3$

$\therefore$ Total cost of raw material is given by

$$Z = 0.85x_1 + 0.9x_2 + 0.5x_3$$

and the objective is to minimize it.

Total iron metal required is 4 tonnes *i.e.*, 4000 kg

$$\Rightarrow \quad x_1 + x_2 + x_3 = 4000$$

The iron melt is to have a minimum of 3.2% carbon

$$\Rightarrow \quad 0.4x_1 + 3.8x_2 + 3.5x_3 \geq 3.2 \times 4000 \quad i.e., \quad 12800$$

The iron melt is to have a maximum of 3.4% carbon

$$\Rightarrow \quad 0.4x_1 + 3.8x_2 + 3.5x_3 \leq 3.4 \times 4000 \quad i.e., \quad 13600$$

The iron melt is to have a minimum of 2.25% silicon

$$\Rightarrow \quad 0.15x_1 + 2.4x_2 + 2.3x_3 \geq 2.25 \times 4000 \quad i.e., \quad 9000$$

The iron melt is to have a maximum of 2.35% silicon

$$\Rightarrow \quad 0.15x_1 + 2.4x_2 + 2.3x_3 \leq 2.35 \times 4000 \quad i.e., \quad 9400$$

Since the amounts of raw material cannot be negative

$$\therefore \quad x_1 \geq 0, \quad x_2 \geq 0, \quad x_3 \geq 0$$

Hence the LP model of the given problem is:

Minimize $\quad Z = 0.85x_1 + 0.9x_2 + 0.5x_3$

subject to the constraints

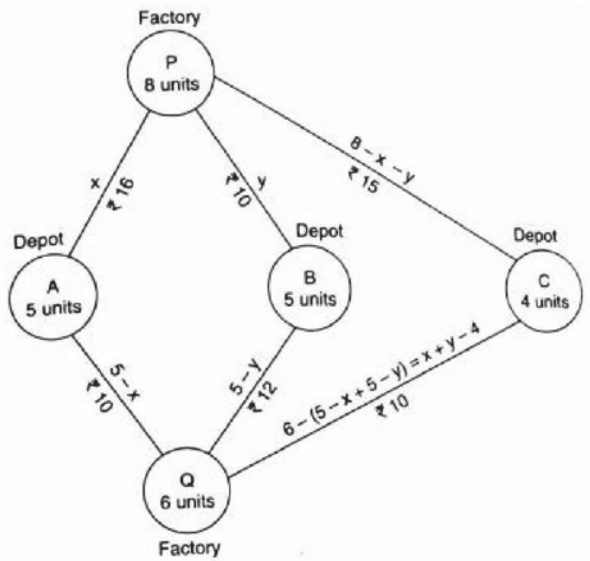
$$x_1 + x_2 + x_3 = 4000$$
$$0.4x_1 + 3.8x_2 + 3.5x_3 \geq 12800$$
$$0.4x_1 + 3.8x_2 + 3.5x_3 \leq 13600$$
$$0.15x_1 + 2.4x_2 + 2.3x_3 \geq 9000$$
$$0.15x_1 + 2.4x_2 + 2.3x_3 \leq 9400$$
$$x_1, x_2, x_3 \geq 0.$$

Example 6. *There is a factory located at each of the two places P and Q. From these locations, a certain commodity is delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are, respectively, 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are, respectively, 8 and 6 units. The cost of transportation per unit is given below:*

To / From	Cost (₹/unit)		
	A	B	C
P	16	10	15
Q	10	12	10

How many units should be transported from each factory to each depot in order that the transportation cost is minimum? Formulate the above linear programming problem mathematically.

Sol. Let x units and y units of the commodity be transported from the factory P to the depots A and B respectively. Since the factory P has the capacity of producing 8 units of commodity, therefore, $(8 - x - y)$ units will be transported from the factory P to the depot C.



Clearly $\quad x \geq 0, \; y \geq 0$ and $8 - x - y \geq 0$ *i.e.*, $x + y \leq 8$.

Now, the weekly requirement of the depot A is 5 units of the commodity and x units are already transported from the factory P, therefore, the remaining $(5 - x)$ units are to be transported from the factory Q. Similarly $(5 - y)$ and $6 - (5 - x + 5 - y) = x + y - 4$ units are to be transported from the factory Q to the depots B and C respectively.

Clearly $\quad 5 - x \geq 0, \; 5 - y \geq 0$ and $x + y - 4 \geq 0$

i.e., $\quad x \leq 5, \; y \leq 5$ and $x + y \geq 4$.

The total transportation cost is given by

$$Z_1 = 16x + 10y + 15(8 - x - y) + 10(5 - x) + 12(5 - y) + 10(x + y - 4)$$

or $\quad Z_1 = x - 7y + 190.$

The objective is to minimize $Z = x - 7y$ since 190 is a constant and does not affect the optimal solution.

Hence the above LPP can be stated mathematically as follows:

Minimize $\quad Z = x - 7y$

subject to the constraints

$$x + y \leq 8$$
$$x \leq 5$$
$$y \leq 5$$
$$x + y \geq 4$$
$$x \geq 0, \quad y \geq 0.$$

EXERCISE 7.1

1. A company produces two types of models M_1 and M_2. Each M_1 model requires 4 hours of grinding and 2 hours of polishing whereas each M_2 model requires 2 hours of grinding and 5 hours of polishing. The company has 2 grinders and 3 polishers. Each grinder works for 40 hours a week and each polisher works for 60 hours a week. Profit on an M_1 model is ₹ 3 and on an M_2 model is ₹ 4. Whatever is produced in a week is sold in the market. How should the company allocate its production capacity to the two types of models so that it may make the maximum profit in a week? Formulate the problem as an LPP.

2. A housewife wishes to mix two types of foods X and Y in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food X costs ₹ 60 per kg and food Y costs ₹ 80 per kg. Food X contains 3 units per kg of vitamin A and 5 units per kg of vitamin B while food Y contains 4 units per kg of vitamin A and 2 units per kg of vitamin B. Formulate the above problem as an LPP to minimize the cost of mixture. *(M.D.U. Dec. 2006)*

3. A manufacturer has 3 machines installed in his factory. Machines I and II are capable of being operated for at most 12 hours whereas machine III must be operated at least 5 hours a day. He produces only two items A and B each requiring the use of the three machines. The number of hours required for producing 1 unit each of the items on the three machines is given in the following table:

Item	*Number of hours required by the machine*		
	I	*II*	*III*
A	1	2	1
B	2	1	$\frac{5}{4}$

He makes a profit of ₹ 6 on item A and ₹ 4 on item B. Assuming that he can sell all he produces, how many of each item should he produce so as to maximize his profit. Formulate this LPP mathematically.

4. An aeroplane can carry a maximum of 200 passengers. A profit of ₹ 400 is made on each first class ticket and a profit of ₹ 300 is made on each economy class ticket. The airline reserves at least 20 seats for first class. However, at least 4 times as many passengers prefer to travel by economy class than by the first class. How many tickets of each class must be sold in order to maximize profit for the airline? Formulate the problem as an LP model. *(M.D.U. Dec. 2005)*

5. Sudha wants to invest ₹ 12000 in Saving Certificates and in National Saving Bonds. According to rules, she has to invest at least ₹ 1000 in Saving Certificates and at least ₹ 2000 in National Saving Bonds. If the rate of interest on Saving Certificates is 8% p.a. and the rate of interest on the National Saving Bonds is 10% p.a., how should she invest her money to earn maximum yearly income. Formulate the problem as an LP model.

6. A firm manufactures 3 products A, B and C. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has two machines M_1 and M_2 and below is the required processing time in minutes for each machine on each product.

Machine	*Product*		
	A	*B*	*C*
M_1	4	3	5
M_2	2	2	4

Machines M_1 and M_2 have 2000 and 2500 machine-minutes respectively. The firm must manufacture 100 A's, 200 B's and 50 C's but not more than 150 A's. Set up an LPP to maximize profit.

7. A firm manufactures headache pills in two sizes A and B. Size A contains 2 grains of asprin, 5 grains of bicarbonate and 1 grain of codeine. Size B contains 1 grain of asprin, 8 grains of bicarbonate and 6 grains of codeine. It is found by users that it requires at least 12 grains of asprin, 74 grains of bicarbonate and 24 grains of codeine for providing immediate effect. It is required to determine the least number of pills a patient should take to get immediate relief. Formulate the above LPP mathematically.

8. The manager of an oil refinery must decide on the optimal mix of two possible blending processes of which the inputs and outputs per production run are as follows:

Processes	*Inputs (units)*		*Outputs (units)*	
	Crude 1	*Crude 2*	*Petrol (superior)*	*Petrol (ordinary)*
A	10	6	10	16
B	12	15	12	12

The availability of the two varieties of crude is limited to the extent of 400 units and 450 units respectively per day. The market demand indicates that at least 200 units and 240 units of the superior and ordinary quality petrol is required every day. The profitability analysis indicate that process A contributes ₹ 480 per run while the process B contributes ₹ 400 per run. The manager is interested in determining an optimal product-mix for maximizing the company's profits. Formulate the problem as an LP model.

9. A tape recorder company manufactures models A, B and C which have profit contributions per unit of ₹ 15, ₹ 40 and ₹ 60 respectively. The weekly minimum production requirements are 25 units, 130 units and 55 units for models A, B and C respectively. Each type of recorder requires a certain amount of time for the manufacturing of component parts, for assembling and for packing. Specifically, a dozen units of model A require 4 hours for manufacturing, 3 hours for assembling and 1 hour for packing. The corresponding figures for a dozen units of model B are 2.5, 4 and 2 and for a dozen units of model C are 6, 9 and 4. During the forthcoming week, the company has available 130 hours of manufacturing, 170 hours of assembling and 52 hours of packing time. Formulate this problem as an LPP model so as to maximize total profit to the company.

10. ABC Foods Company is developing a low-calorie high-portein diet supplement called Hi-Pro. The specifications for Hi-Pro have been established by a panel of medical experts. These specifications along with the calorie, protein and vitamin content of three basic foods, are given in the following table:

Nutritional Elements	*Units of Nutritional Elements (Per 100 gm Serving of Basic Foods)*			*Basic Foods Hi-Pro Specifications*
	1	*2*	*3*	
Calories	350	250	200	300
Proteins	250	300	150	200
Vitamin A	100	150	75	100
Vitamin C	75	125	150	100
Cost per serving (₹)	1.50	2.00	1.20	

What quantities of foods 1, 2 and 3 should be used ? Formulate this problem as an LP model to minimize cost of serving.

11. A firm manufactures two items A and B. It purchases castings which are then machined, bored and polished. Castings for items A and B cost ₹ 3 and ₹ 4 each and are sold at ₹ 6 and ₹ 7 respectively. Running costs of these machines are ₹ 20, ₹ 14 and ₹ 17.50 per hour respectively.

Formulate the problem so that the product mix maximizes the profit. The capacities of the machines are

	Item–A	Item–B
Machining	25 per hr.	40 per hr.
Boring	28 per hr.	35 per hr.
Polishing	35 per hr.	25 per hr.

(K.U.K. Dec. 2010)

12. A brick manufacturer has two depots, A and B, with stocks of 30000 and 20000 bricks respectively. He receives orders from three building's P, Q and R for 15000, 20000 and 15000 bricks respectively. The cost of transporting 1000 bricks to the builders from the depots (in ₹) are given below:

To / From	*Transportation cost per 1000 bricks (in ₹)*		
	P	*Q*	*R*
A	40	20	20
B	20	60	40

How should the manufacturer fulfil the orders so as to keep the cost of transportation minimum? Formulate the above LPP mathematically.

Answers

1. Maximize $Z = 3x + 4y$
subject to the constraints $2x + y \le 40$
$2x + 5y \le 180$
$x \ge 0, \; y \ge 0.$

2. Minimize $Z = 60x + 80y$
subject to the constraints $3x + 4y \ge 8$
$5x + 2y \ge 11$
$x \ge 0, \; y \ge 0.$

3. Maximize $Z = 6x + 4y$
subject to the constraints $x + 2y \le 12$
$2x + y \le 12$
$4x + 5y \ge 20$
$x \ge 0, \; y \ge 0.$

4. Maximize $Z = 400x + 300y$
subject to the constraints $x + y \le 200$
$x \ge 20, \; y \ge 4x$
$x \ge 0, \; y \ge 0.$

5. Maximize $Z = \frac{8}{100} x + \frac{10}{100} y$
subject to the constraints $x + y \le 12000$
$x \ge 1000, \; y \ge 2000$
$x \ge 0, \; y \ge 0.$

6. Maximize $Z = 3x_1 + 2x_2 + 4x_3$
subject to the constraints
$4x_1 + 3x_2 + 5x_3 \le 2000$
$2x_1 + 2x_2 + 4x_3 \le 2500$
$100 \le x_1 \le 150$
$x_2 \ge 200, \; x_3 \ge 50.$

7. Minimize $Z = x + y$
subject to the constraints
$2x + y \ge 12$
$5x + 8y \ge 74$
$x + 6y \ge 24$
$x \ge 0, y \ge 0.$

8. Maximize $Z = 480x + 400y$
subject to the constraints $10x + 12y \le 400$
$6x + 15y \le 450$
$10x + 12y \ge 200$
$16x + 12y \ge 240$
$x \ge 0, \; y \ge 0.$

9. Maximize $Z = 15x_1 + 40x_2 + 60x_3$
subject to the constraints
$x_1 \ge 25$
$x_2 \ge 130$
$x_3 \ge 55$
$4x_1 + 2.5x_2 + 6x_3 \le 1560$
$3x_1 + 4x_2 + 9x_3 \le 2040$
$x_1 + 2x_2 + 4x_3 \le 624.$

10. Minimize $Z = 1.5x_1 + 2x_2 + 1.2x_3$
subject to the constraints
$350x_1 + 250x_2 + 200x_3 \ge 300$
$250x_1 + 300x_2 + 150x_3 \ge 200$
$100x_1 + 150x_2 + 75x_3 \ge 100$
$75x_1 + 125x_2 + 150x_3 \ge 100$
$x_1, x_2, x_3 \ge 0.$

11. Maximize $Z = 1.2x + 1.4y$
subject to the constraints
$40x + 25y \le 1000$
$35x + 28y \le 980$
$25x + 35y \le 875$
$x, y \ge 0.$

12. Minimize $Z = 40x - 20y$
subject to the constraints
$x + y \ge 15$
$x \le 15$
$y \le 20$
$x + y \le 30$
$x \ge 0, y \ge 0$

where 1 unit of bricks = 1000 bricks
Total cost of transportation is $Z_1 = Z + 1500.$

7.4. GRAPH OF A LINEAR INEQUALITY

The constraints in the mathematical model of an LPP are in the form of linear inequalities. Let us see how to graph linear inequalities involving two variables.

A linear inequality in two variables x and y is of the form $ax + by < c$ or $ax + by \le c$ or $ax + by > c$ or $ax + by \ge c$, where at least one of a and b is non-zero. The graph of a linear inequality in two variables x and y is the set of all ordered pairs (x, y) for which the inequality holds. The following three steps are involved in graphing a linear inequality.

1. Graph the corresponding linear equation $ax + by = c$ which is a straight line. Find any two distinct points on the line and draw a straight line through them. (For convenience, choose one point on x-axis by putting $y = 0$ and the other on y-axis by putting $x = 0$.) The line divides the x-y plane into two half planes.

2. To decide which of the two half-planes satisfies the linear inequality, choose a point P not on the line. (If the line does not pass through the origin, then origin is the best choice). Check whether P satisfies the inequality.

3. If the coordinates of P satisfy the inequality, then every point in the half plane containing P satisfies the inequality. If the coordinates of P donot satisfy the inequality, then every point in the other half plane (not containing P) satisfies the inequality.

Remark. $x = 0$ is the equation of y-axis. $x = a$, $a > 0$ is a straight line parallel to y-axis and at a distance 'a' to its right. $x > a$ is the half plane to the right of the line $x = a$ and $x < a$ is the half plane to the left of the line $x = a$.

$x = -a$, $a > 0$ is a straight line parallel to y-axis and at a distance 'a' to its left. $x > -a$ is the half plane to the right of the line $x = -a$ and $x < -a$ is the half plane to the left of the line $x = -a$.

Similarly, $y = 0$ is the equation of x-axis. $y = a$ and $y = -a$, $a > 0$, are lines parallel to x-axis at a distance 'a' above and below the x-axis respectively. $y > a$ is the half plane above the line $y = a$ and $y < a$ is the half plane below the line $y = a$. Similar remarks apply to $y > -a$ and $y < -a$.

ILLUSTRATIVE EXAMPLES

Example 1. *Graph the linear inequality:*

$$3x + 4y \le 12.$$

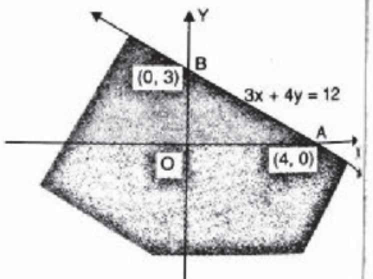
Sol. The given inequation is $3x + 4y \le 12$ ...(1)

Replacing $\le$ by $=$, the corresponding equation is $3x + 4y = 12$ ...(2)

Putting $y = 0$ in (2), $3x = 12$ or $x = 4$

Putting $x = 0$ in (2), $4y = 12$ or $y = 3$

$\therefore$ The graph of line (2) passes through the points A(4, 0) and B(0, 3). The line through these two points in the graph of equation (2). This line divides the plane into two half-planes.



Putting $x = 0$ and $y = 0$ in (1), we get $0 \le 12$ which is true. Therefore the origin O(0, 0) lies in the feasible region. Hence, the shaded region below the line AB and the points on the line AB (as shown in the adjoining figure) constitute the graph of the inequality (1).

Example 2. *Solve graphically the following system of linear inequations:*

$$x \ge 4, y \ge 2.$$

Sol. The given system of inequations is

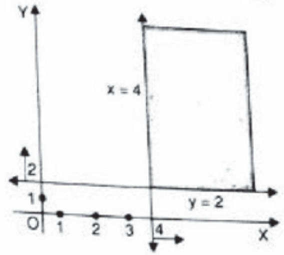
$$x \ge 4 \quad ...(1), \qquad y \ge 2 \quad ...(2)$$

The equation corresponding to inequation (1) is $x = 4$. This is a line parallel to y-axis and at a distance 4 units to the **right** of y-axis.

Putting $x = 0$, $y = 0$ in (1), $0 \ge 4$ which is not true.

$\therefore$ Graph of inequation (1) does not contain origin *i.e.*, $x \ge 4$ represents the region away from the origin.

$\therefore$ The inequation (1) represents the half-plane to the right of this line, including this line.



The equation corresponding to inequation (2) is $y = 2$. This is a line parallel to x-axis and at a distance 2 units above it.

Putting $x = 0$ and $y = 0$ in (2), $0 \ge 2$ which is not true.

$\therefore$ Graph of inequation (2) does not contain the origin.

$\therefore$ The inequation (2) represents the half-plane **above** this line, including this line.

Hence the common region is the shaded region. Any point in this shaded region represents a solution of the given system of inequations.

Example 3. *Solve graphically* $x + 3y \le 12$, $3x + y \le 12$, $x \ge 0$, $y \ge 0$.

Sol. The given system of inequations is

$$x + 3y \le 12 \quad ...(1)$$
$$3x + y \le 12 \quad ...(2)$$
and
$$x \ge 0, y \ge 0 \quad ...(3)$$

The equation corresponding to inequation (1) is $x + 3y = 12$

Putting $x = 0$, $3y = 12$ or $y = 4$

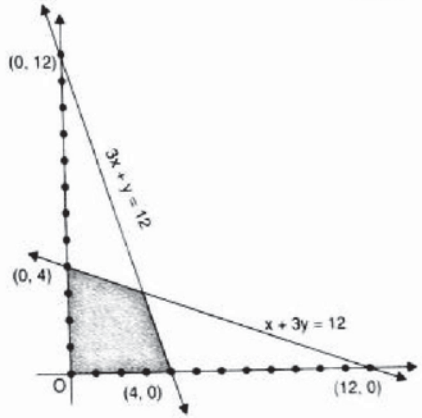
Putting $y = 0$, $x = 12$

$\therefore$ Graph of $x + 3y = 12$ is the straight line through the points (0, 4) and (12, 0).

Putting $x = 0$ and $y = 0$ in (1); $0 \le 12$ which is true.

$\therefore$ Graph of inequation (1) contains the origin.

$\therefore$ The inequation (1) represents the half plane on the origin side of this line, including the line.



The equation corresponding to inequation (2) is $3x + y = 12$

Putting $x = 0$, $y = 12$

Putting $y = 0$, $3x = 12$ or $x = 4$

$\therefore$ Graph of $3x + y = 12$ is the straight line through the points (0, 12) and (4, 0).

Putting $x = 0$ and $y = 0$ in (2), $0 \le 12$ which is also true.

$\therefore$ Graph of inequation (2) also contains the origin.

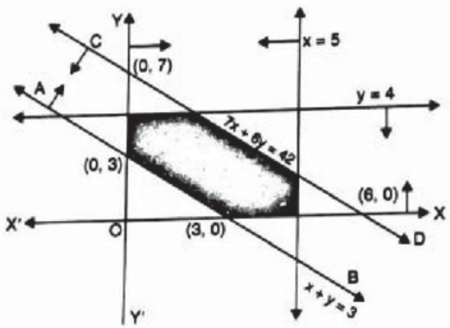
$\therefore$ The inequation (2) represents the half-plane on the origin side of this line, including the line.

Inequation (3) is $x \ge 0$, $y \ge 0$ $\Rightarrow$ feasible region is in first quadrant only.

$\therefore$ The common feasible region (*i.e.*, intersection of all feasible regions) in first quadrant is shown as shaded in the figure. Every point in this shaded region represents a solution of the given system of linear inequations.

Example 4. *Solve graphically: $x + y \geq 3$, $7x + 6y \leq 42$, $x \leq 5$, $y \leq 4$, $x \geq 0$, $y \geq 0$.*

Sol. The graph of $x + y = 3$ is the line AB through the points (3, 0) and (0, 3). Since (0, 0) does not satisfy $x + y \geq 3$, the region represented by this inequation is the half-plane above the line AB, including this line.



The graph of $7x + 6y = 42$ is the line CD through the points (6, 0) and (0, 7) (By putting $x = 0$ and $y = 0$ respectively in $7x + 6y = 42$). Since (0, 0) satisfies $7x + 6y \leq 42$, the region represented by this inequation is the half-plane on the origin side of line CD, including this line.

$x \leq 5$ represents the half-plane on the left of the line $x = 5$, including this line.

$y \leq 4$ represents the half-plane below the line $y = 4$, including this line.

$x \geq 0$ and $y \geq 0$ $\Rightarrow$ Feasible region is in first quadrant only.

The common feasible region (*i.e.*, intersection of all feasible regions) is shown shaded in the figure. Every point in this shaded region represents a solution of the given system of linear inequations.

Example 5. *Solve graphically the inequations $3x + 2y \leq 24$, $x + 2y \leq 16$, $x + y \leq 10$, $x \geq 0$, $y \geq 0$.*

Sol. The given system of inequations is

$$3x + 2y \leq 24 \quad \ldots(1)$$

$$x + 2y \leq 16 \quad \ldots(2)$$

$$x + y \leq 10 \quad \ldots(3)$$

and

$$x \geq 0, y \geq 0 \quad \ldots(4)$$

The equation corresponding to inequation (1) is $3x + 2y = 24$.

Putting $x = 0$, $2y = 24$ or $y = 12$

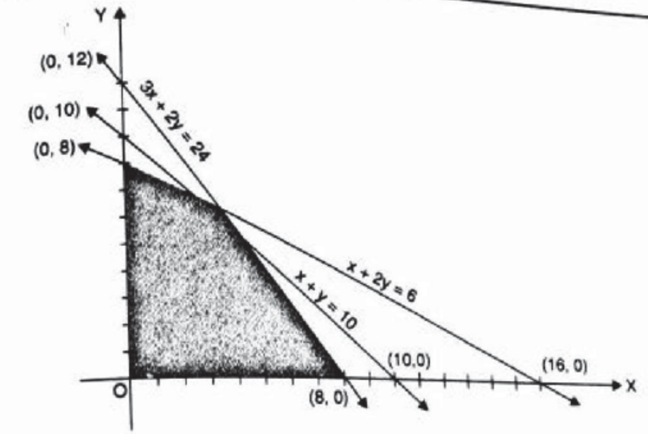
Putting $y = 0$, $3x = 24$ or $x = 8$.

$\therefore$ Graph of $3x + 2y = 24$ is the straight line through the points (0, 12) and (8, 0).

Putting $x = 0$ and $y = 0$ in (1), $0 \leq 24$ which is true.

$\therefore$ Graph of inequation (1) contains the origin.

$\therefore$ The inequation (1) represents the half-plane on the origin side of this line, including the line.



The equation corresponding to inequation (2) is

$$x + 2y = 16$$

Putting $x = 0$, $2y = 16$ or $y = 8$

Putting $y = 0$, $x = 16$

$\therefore$ Graph of $x + 2y = 16$ is the straight line through the points (0, 8) and (16, 0).

Putting $x = 0$ and $y = 0$ in (2), $0 \leq 16$ which is true.

$\therefore$ Graph of inequation (2) also contains the origin.

$\therefore$ The inequation (2) represents the half plane on the origin side of this line, including the line.

The equation corresponding to inequation (3) is

$$x + y = 10$$

Putting $x = 0$, $y = 10$

Putting $y = 0$, $x = 10$

$\therefore$ Graph of $x + y = 10$ is the straight line through the points (0, 10) and (10, 0).

Putting $x = 0$ and $y = 0$ in (3), $0 \leq 10$ which is also true.

$\therefore$ Graph of inequation (3) also contains the origin.

$\therefore$ The inequation (3) represents the half-plane on the origin side of this line, including the line.

Inequation (4) is $x \geq 0, y \geq 0$ $\Rightarrow$ Feasible region is in first quadrant only.

$\therefore$ The common feasible region (*i.e.*, intersection of all feasible regions) in first quadrant is shown as shaded in the figure. Every point in this shaded region represents a solution of the given system of linear inequations.

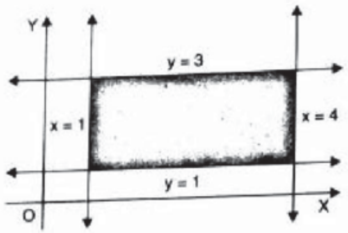
EXERCISE 7.2

Graph the following systems of inequations and shade the feasible region:

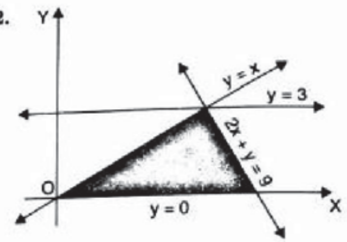
1. $x \geq 1, x \leq 4, y \geq 1, y \leq 3$ (or $1 \leq x \leq 4, 1 \leq y \leq 3$).
2. $0 \leq y \leq 3,\ y \leq x, 2x + y \leq 9$.
3. $3x + 2y \leq 18, x + 2y \leq 10, x \geq 0, y \geq 0$.
4. $x + y \leq 9, y \geq x, x \geq 1$.
5. $3x + 4y \leq 60, x + 3y \leq 30, x \geq 0, y \geq 0$
6. $2x + y \leq 24, x + y \leq 11, 2x + 5y \leq 40, x \geq 0, y \geq 0$.

Answers

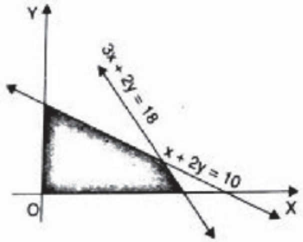
1.



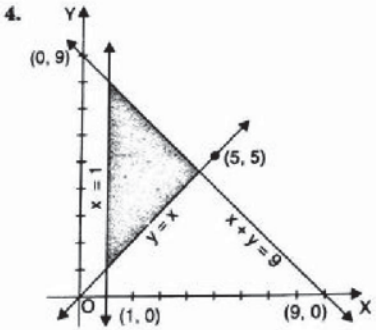
2.



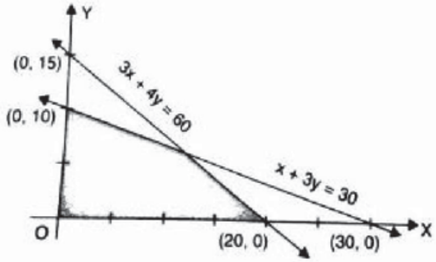
3.



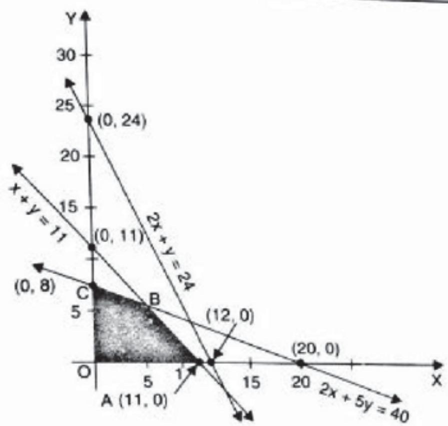
4.



5.



6.



7.5. THE GRAPHICAL METHOD OF SOLVING AN LPP

For linear programming problems having only two variables, the set of all feasible solutions can be displayed graphically by determining the feasible region. The points lying within the feasible region satisfy all the constraints. The graphical approach gives an insight into the basic concepts and provides valuable understanding for solving LP problems involving more than two variables algebraically. Problems involving more than two variables cannot be solved graphically.

There are two techniques of solving an LPP by graphical method

(*i*) Corner point method.

(*ii*) Iso-profit or Iso-cost method.

We shall discuss only the first technique.

CORNER POINT METHOD

This method is based on a theorem called **extreme point theorem.**

Extreme Point Theorem. The optimal solution to a linear programming problem, if it exists, occurs at an extreme point (corner) of the feasible region.

The collection of all feasible solutions to an LPP constitutes a convex set whose extreme points correspond to the basic feasible solutions.

Working procedure to solve an LPP graphically:

1. Formulate the given problem as an LPP.
2. Plot the constraints and shade the common region that satisfies all the constraints simultaneously. The shaded area is called the feasible region.
3. Determine the coordinates of each corner of the feasible region.
4. Find the value of the objective function $Z = ax + by$ at each corner point. Let M and m denote respectively the largest and the smallest values of Z at the corner points.

5. When the feasible region is **bounded**, M and m are the maximum and the minimum values of Z.
6. When the feasible region is **unbounded**, we have:

(a) M is the maximum value of Z, if the open half plane determined by Z > M *i.e.*, $ax + by > M$ has no point in common with the feasible region. Otherwise Z has no maximum value.

(b) m is the minimum value of Z, if the open half plane determined by Z < m *i.e.*, $ax + by < m$ has no point in common with the feasible region. Otherwise Z has no minimum value.

ILLUSTRATIVE EXAMPLES

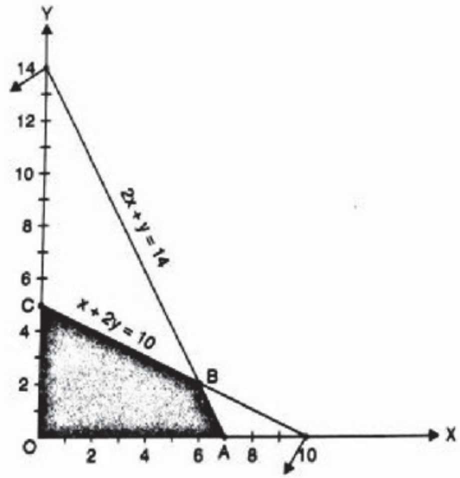
Example 1. *Solve the following linear programming problem graphically:*

Maximize $Z = 2x + 3y$

subject to the constraints

$$x + 2y \le 10$$
$$2x + y \le 14$$
$$x, y \ge 0.$$

Sol. Mathematical formulation of LPP is already given. The non-negativity constraints $x \ge 0, y \ge 0$ imply that the feasible region is in first quadrant only. Plot each constraint by first treating it as a linear equation and then using the inequality condition of each constraint, mark the feasible region as shown in the figure. The feasible region is shown by the shaded area OABC in the figure.



Since the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates.

Clearly O = (0, 0), A = (7, 0), B = (6, 2), C = (0, 5)

Now we find the value of the objective function Z = 2x + 3y at each corner point.

Corner Point	Coordinates (x, y)	Value of Objective Function $Z = 2x + 3y$
O	(0, 0)	0 + 0 = 0
A	(7, 0)	2(7) + 3(0) = 14
B	(6, 2)	2(6) + 3(2) = $\boxed{18}$
C	(0, 5)	2(0) + 3(5) = 15

The maximum value of Z is 18 at B(6, 2). Hence the optimal solution to the given LP problem is

$$x = 6, \quad y = 2 \quad \text{and Max. } Z = 18$$

Remark. The coordinates of B are obtained by solving $x + 2y = 10$ and $2x + y = 14$.

Example 2. *Solve the following LP problem by the graphical method:*

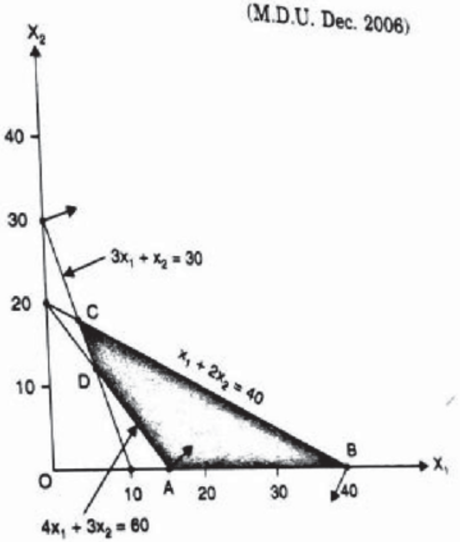
Minimize $Z = 20x_1 + 10x_2$

subject to the constraints

$$x_1 + 2x_2 \le 40$$
$$3x_1 + x_2 \ge 30$$
$$4x_1 + 3x_2 \ge 60$$
$$x_1, x_2 \ge 0$$

(M.D.U. Dec. 2006)

Sol. Mathematical formulation of LPP is already given. The non-negativity constraints $x_1, x_2 \ge 0$ imply that the feasible region is in first quadrant only. Plot each constraint by first treating it as a linear equation and then using the inequality condition of each constraint, mark the feasible region as shown in the figure. The feasible region is shown by the shaded area ABCD in the figure.



Since the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates.

Clearly A = (15, 0), B = (40, 0)
C = (5, 18), D = (6, 12)

Now we find the value of the objective function $Z = 20x_1 + 10x_2$ at each corner point.

Corner Point	Coordinates (x_1, x_2)	Value of Objective Function $Z = 20x_1 + 10x_2$
A	(15, 0)	20(15) + 10(0) = 300
B	(40, 0)	20(40) + 10(0) = 800
C	(5, 18)	20(5) + 10(18) = 280
D	(6, 12)	20(6) + 10(12) = $\boxed{240}$

The minimum value of Z is 240 at D(6, 12). Hence the optimal solution to the given LP problem is

$$x_1 = 6, \quad x_2 = 12 \quad \text{and Min. } Z = 240.$$

Example 3. *A manufacturer has 3 machines installed in his factory. Machines I and II are capable of being operated for at the most 12 hours, whereas machine III must be operated at least 5 hours a day. He produces only two items, each requiring the use of the three machines.*

The number of hours required for producing 1 unit of each of the items A and B on the three machines are given in the following table:

Items	*Number of hours required on the machines*		
	I	*II*	*III*
A	*1*	*2*	*1*
B	*2*	*1*	$\frac{5}{4}$

He makes a profit of ₹60 on item A and ₹40 on item B. Assuming that he can sell all that he produces, how many of each item should be produce so as to maximize his profit. Solve the LP problem graphically.

Sol. Suppose the manufacturer produces x and y units of items A and B respectively. His objective is to maximize the total profit = ₹ $(60x + 40y)$

$\therefore$ Objective function is given by $Z = 60x + 40y$

Machine hour constraints are:

Machine I $\quad x + 2y \le 12$

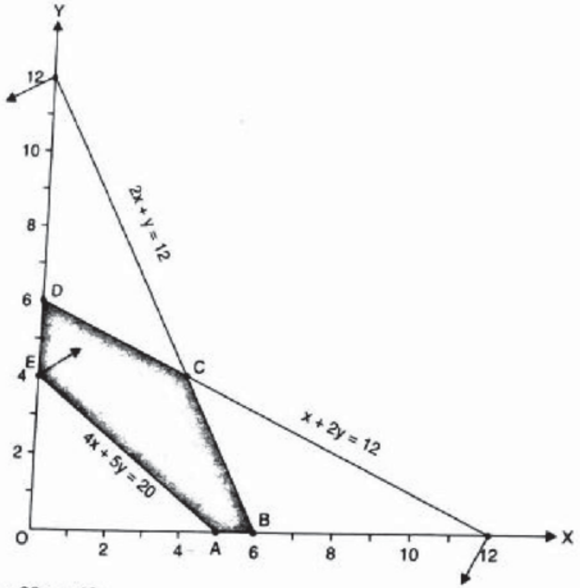
Machine II $\quad 2x + y \le 12$

Machine III $\quad x + \frac{5}{4}y \ge 5$ or $4x + 5y \ge 20$

Non-negativity constraints are $\quad x, y \ge 0$

(since it makes no sense to assign negative values to x and y).

Thus, the mathematical formulation of the LP problem is:



Maximize $Z = 60x + 40y$

subject to the constraints

$$x + 2y \le 12$$
$$2x + y \le 12$$
$$4x + 5y \ge 20$$
$$x, y \ge 0$$

The feasible region is shown by the shaded area ABCDE in the figure.

Since the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates

Here $A = (5, 0)$, $B = (6, 0)$, $C = (4, 4)$, $D = (0, 6)$, $E = (0, 4)$

Now we find the value of the objective function $Z = 60x + 40y$ at each corner point.

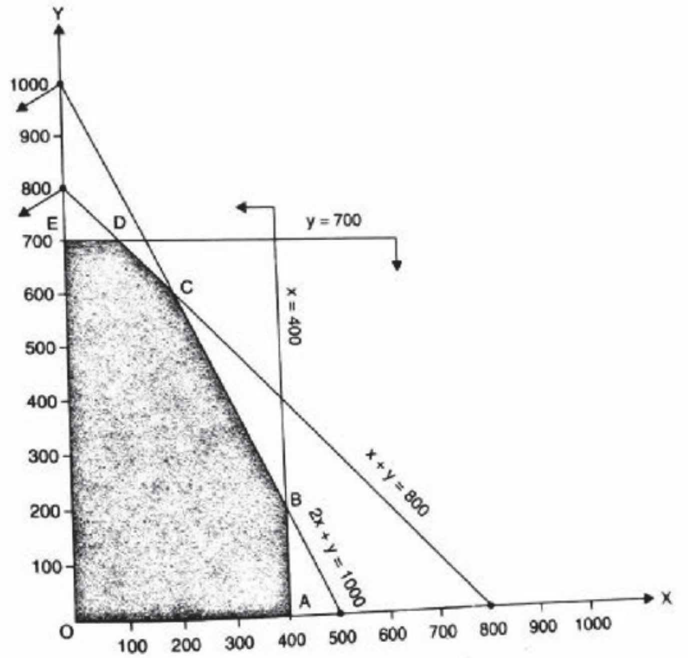
Corner Point	*Coordinates* (x, y)	*Value of Objective Function* $Z = 60x + 40y$
A	(5, 0)	$60(5) + 40(0) = 300$
B	(6, 0)	$60(6) + 40(0) = 360$
C	(4, 4)	$60(4) + 40(4) = \boxed{400}$
D	(0, 6)	$60(0) + 40(6) = 240$
E	(0, 4)	$60(0) + 40(4) = 160$

The maximum value of Z is 400 at C(4, 4).

$\therefore$ The manufacturer should produce $x = 4$ units of item A and $y = 4$ units of item B to get the maximum profit of Rs. 400.

Example 4. *A company makes two kinds of leather belts, A and B. Belt A is a high quality belt and B is of lower quality. The respective profits are ₹4 and ₹3 per belt. Each belt of type A requires twice as much time as a belt of type B, and, if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day available for belt B. What should be the daily production of each type of belt?*

Sol. Let x and y denote respectively the number of belts of type A and type B produced per day.



The objective is to maximize the profit Z (in rupees) given by

$$Z = 4x + 3y$$

Since the rate of producing type B belts is 1000 per day, the total time taken to produce y belts of type B is $\dfrac{y}{1000}$.

Also, each belt of type A requires twice as much time as a belt of type B, the rate of producing type A belts is 500 per day and the total time taken to produce x belts of type A is $\dfrac{x}{500}$.

$\therefore$ The time constraint is $\dfrac{x}{500} + \dfrac{y}{1000} \le 1$

or $$2x + y \le 1000$$

The constraint imposed by supply of leather is

$$x + y \le 800$$

The constraint imposed by supply of fancy buckles is

$$x \le 400$$

The constraint imposed by supply of buckles for type B belts is

$$y \le 700$$

Since the number of belts cannot be negative, we have non-negativity constraints

$$x \ge 0, \quad y \ge 0$$

$\therefore$ The mathematical formulation of the problem is:

Maximize $\quad Z = 4x + 3y$

subject to the constraints
$$2x + y \le 1000$$
$$x + y \le 800$$
$$x \le 400$$
$$y \le 700$$
$$x, y \ge 0$$

The feasible region is shown by the shaded area OABCDE in the figure.

Since the optimal value of the objective function occurs at one of the corners of the feasible region, we determine their coordinates.

Here O = (0, 0), A = (400, 0), B = (400, 200), C = (200, 600), D = (100, 700), E = (0, 700)

Now we find the value of the objective function Z = 4x + 3y at each corner point.

Corner Point	*Coordinates* (x, y)	*Value of Objective Function* $Z = 4x + 3y$
O	(0, 0)	4(0) + 3(0) = 0
A	(400, 0)	4(400) + 3(0) = 1600
B	(400, 200)	4(400) + 3(200) = 2200
C	(200, 600)	4(200) + 3(600) = $\boxed{2600}$
D	(100, 700)	4(100) + 3(700) = 2500
E	(0, 700)	4(0) + 3(700) = 2100

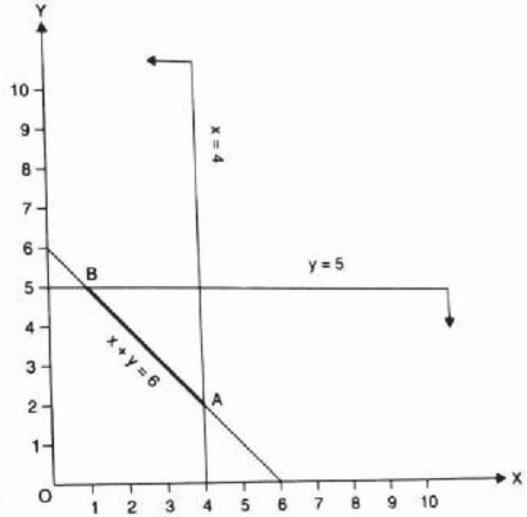
The maximum value of Z is 2600 at C(200, 600).

$\therefore$ The company should produce 200 belts of type A and 600 belts of type B in order to get the maximum profit of ₹ 2600 per day.

Example 5. *Solve the following linear programming problem graphically:*

Minimize $\quad Z = 3x + 5y$

subject to the constraints $\quad x + y = 6$
$$x \le 4$$
$$y \le 5$$
$$x \ge 0, \quad y \ge 0$$



Sol. Here the feasible region of the LPP is the line segment AB with A = (4, 2) and B = (1, 5). These are the corner points of the feasible region.

At A(4, 2), $\quad Z = 3(4) + 5(2) = 22$

At B(1, 5), $\quad Z = 3(1) + 5(5) = 28$

The minimum value of Z is 22 which occurs at A(4, 2).

Hence, optimal solution is $x = 4$, $y = 2$ and optimal value = 22.

Example 6. *A person consumes two types of food, A and B, everyday to obtain 8 units of protein, 12 units of carbohydrates and 9 units of fat which is his daily minimum requirement. 1 kg of food A contains 2, 6, 1 units of protein, carbohydrates and fat, respectively. 1 kg of food B contains 1, 1 and 3 units of protein, carbohydrates and fat, respectively. Food A costs ₹8 per kg while food B costs ₹5 per kg. Form an LPP to find how many kgs of each food should he buy daily to minimize his cost of food and still meet minimal nutritional requirements and solve it.*

Sol. Suppose the person buys x kg of food A and y kg of food B daily. Total cost of food (in rupees) is given by

$$Z = 8x + 5y$$

The objective is to minimize Z.

x kg of food A contains $2x$ units of protein.

y kg of food B contains y units of protein.

Since minimum requirement of protein is 8 units, therefore, protein constraint is

$$2x + y \ge 8$$

Similarly $\quad 6x + y \ge 12 \quad$ (carbohydrate constraint)

$$x + 3y \ge 9 \quad \text{(fat constraint)}$$

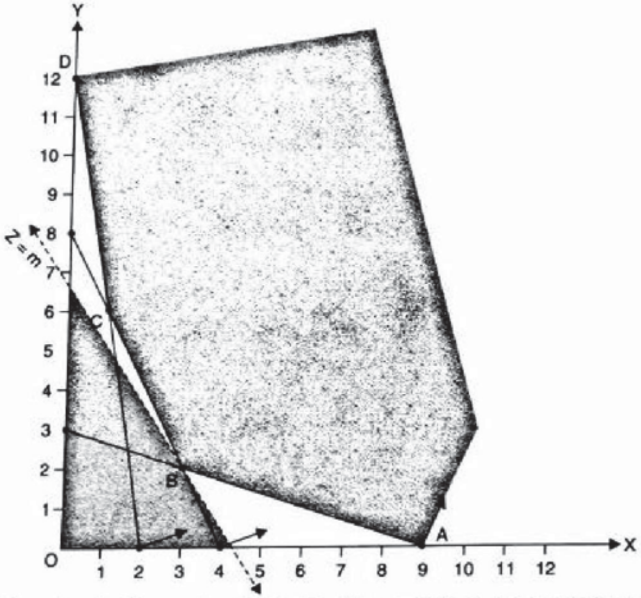
Also $\quad x, y \ge 0$

$\therefore$ The mathematical formulation of the problem is:

Minimize $\quad Z = 8x + 5y$

subject to the constraints
$$2x + y \ge 8$$
$$6x + y \ge 12$$

$$x + 3y \geq 9$$
$$x, y \geq 0$$



The feasible region is shown shaded in the figure. It is unbounded. The extreme points of the feasible region are:

A = (9, 0), B = (3, 2), C = (1, 6), D = (0, 12)

Now we find the value of the objective function $Z = 8x + 5y$ at each corner point.

Corner Point	*Coordinates* (x, y)	*Value of Objective Function* $Z = 8x + 5y$
A	(9, 0)	$8(9) + 5(0) = 72$
B	(3, 2)	$8(3) + 5(2) = \boxed{34} = m$
C	(1, 6)	$8(1) + 5(6) = 38$
D	(0, 12)	$8(0) + 5(12) = 60$

The minimum value of Z is 34 at B(3, 2).

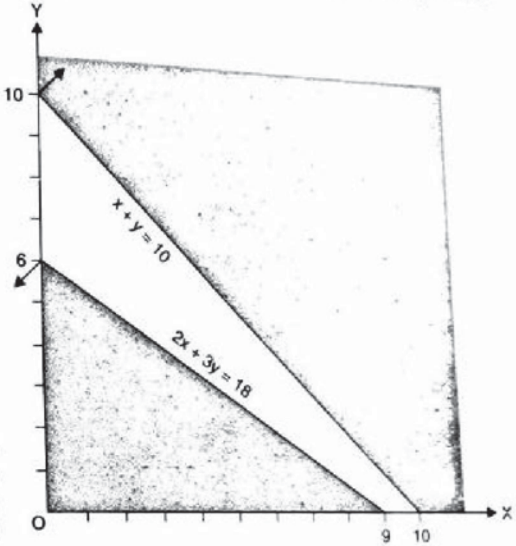
But this value is doubtful. It is yet to be confirmed. Now we draw the graph of $Z < m$ *i.e.,* $8x + 5y < 34$. Since the half plane determined by $Z < m$ has no point in common with the feasible region, $m = 34$ is the minimum value of Z.

$\therefore$ The person should buy 3 kg of food A and 2 kg of food B at the minimum cost of ₹ 34.

7.6. SOME EXCEPTIONAL CASES

So far we have been solving LP problems which may be called 'well behaved' in the sense each problem has a unique optimal solution. However, there are certain exceptional cases which must be taken into consideration.

I. Infeasible solution. Wrong formulation of an LPP results into inconsistent constraints and no value of the variables satisfies all the constraints simultaneously. In such cases, the linear programming problem is said to have no feasible solution or infeasible solution. Infeasibility depends only on the constraints and has nothing to do with the objective function.

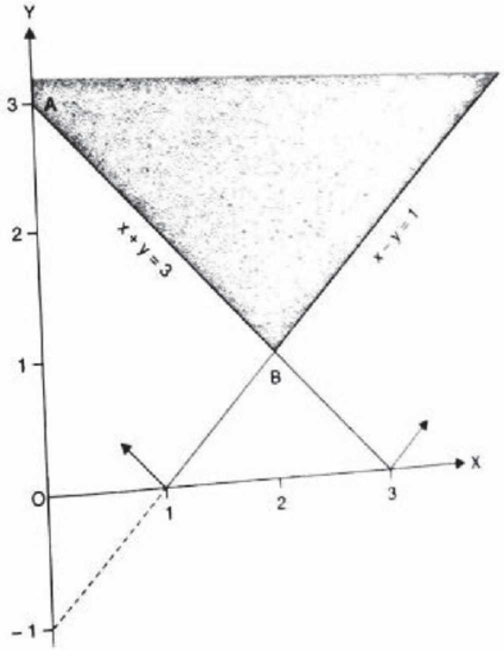


Example. *Maximize* $Z = 5x + 12y$ *subject to the constraints* $2x + 3y \leq 18$

$$x + y \geq 10$$
$$x, y \geq 0.$$

Sol. From the adjoining figure, it is clear that there is no point in the first quadrant satisfying all the constraints. Hence, there is no feasible solution to the problem because of conflicting constraints.

II. Unbounded Solutions. Some linear programming problems have unbounded feasible region so that the variables can take any value in the unbounded region without violating any constraint. If we wish to maximize the objective function Z, then for any value of Z we can find a feasible solution with a greater value of Z. Such problems are said to have unbounded solutions.



Example. *Maximize* $Z = 2x + 5y$ *subject to the constraints* $x + y \geq 3$, $x - y \leq 1$, $x, y \geq 0$.

Sol. The feasible region, shown shaded in the figure, is unbounded. The corner points of the feasible region are A(0, 3) and B(2, 1).

$$Z(A) = 2(0) + 5(3) = 15$$
$$Z(B) = 2(2) + 5(1) = 9$$

Since the given LP problem is of maximization. There are infinite number of points in the feasible region where the value of the objective function is more than this value Z(A) = 15. Both the variables x and y can be made arbitrarily large and there is no limit to the value of Z. Hence the problem has unbounded solutions.

III. Multiple Optimal Solutions. When there exist more than one points in the feasible region such that the objective function Z has the same optimal value, say k, then each such point corresponds to an optimal solution. The LPP has multiple solutions. Each of such optimal solutions is called multiple optimal solution or an alternative optimal solution.

The following two conditions must be satisfied for multiple optimal solutions to exist:

(i) the objective function must be parallel to a constraint which forms the boundary of the feasible region.

(ii) the constraint must form a boundary of the feasible region in the direction of the optimal movement of the objective function.

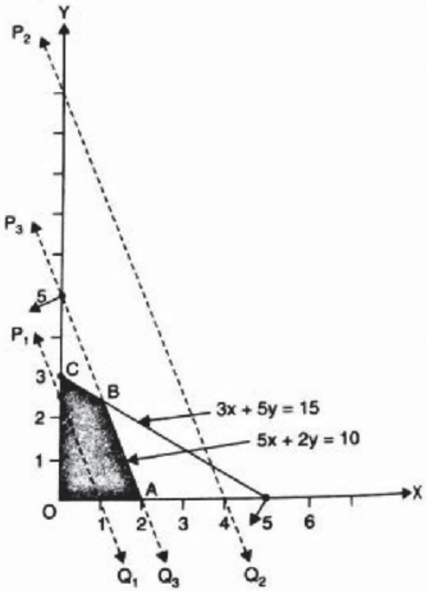
Example. *Solve the following problem graphically.*

Maximize $Z = 2.5x + y$

subject to the constraints $3x + 5y \le 15$

$$5x + 2y \le 10$$

$$x, y \ge 0.$$



Sol. The feasible region is shown by the shaded area OABC in the figure.

We give a constant value 2.5 to Z and draw the line $2.5x + y = 2.5$, shown by P_1Q_1. Give another value 10 to Z and draw the line $2.5x + y = 10$, shown by P_2Q_2. Moving P_1Q_1 parallel to itself towards Z increasing *i.e.,* towards P_2Q_2 as far are possible, until the farthest point B within the feasible region is touched by the line, shown by P_3Q_3. Clearly P_3Q_3 contains the line segment AB which corresponds to the constraint $5x + 2y \le 10$ and forms the boundary of the feasible region. Solving $3x + 5y = 15$ and $5x + 2y = 10$, we have $B = \left(\frac{20}{19}, \frac{45}{19}\right)$ and

$Z(B) = \frac{5}{2}\left(\frac{20}{19}\right) + \frac{45}{19} = 5$. Also $A = (2, 0)$ and $Z(A) = 2.5 \times 2 + 0 = 5$. All points on the line segment AB give the same optimal value Z = 5. The optimal value is unique but there are infinite number of optimal solutions. Every point on AB corresponds to an optimal solution.

Here the objective function has the same slope as the constraint line $5x + 2y = 10$.

IV. Redundancy. A constraint in a given LPP is said to be redundant if the feasible region of the problem is not affected by deleting that constraint.

Example. *Graph the feasible region of the following problem and identify a redundant constraint.*

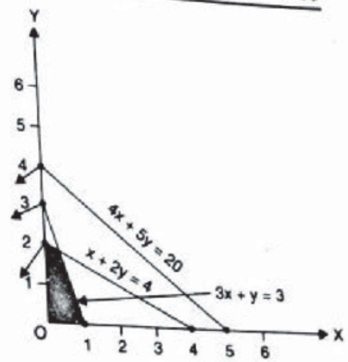
Maximize $Z = 5x + 6y$

subject to the constraints $x + 2y \le 4$

$$4x + 5y \le 20$$

$$3x + y \le 3$$

$$x, y \ge 0.$$



Sol. The feasible region is shown shaded in the figure. Clearly, the constraint $4x + 5y \le 20$ is redundant as it does not affect the feasible region. If this constraint is dropped, the feasible region remains the same.

EXERCISE 7.3

Solve the following LP problems graphically (using corner point method):

1. Maximize $Z = 5x_1 + 7x_2$
   subject to the constraints
   $$x_1 + x_2 \le 4$$
   $$3x_1 + 8x_2 \le 24$$
   $$10x_1 + 7x_2 \le 35$$
   $$x_1, x_2 \ge 0.$$
   (M.D.U. Dec. 2010)

2. Maximize $Z = 2x + 3y$
   subject to the constraints
   $$x + y \le 30$$
   $$3 \le y \le 12$$
   $$x - y \ge 0$$
   $$0 \le x \le 20.$$

3. Maximize $Z = 5x_1 + 3x_2$
   subject to the constraints
   $$3x_1 + 5x_2 \le 15$$
   $$5x_1 + 2x_2 \le 10$$
   $$x_1, x_2 \ge 0.$$
   (M.D.U. May 2009)

4. Minimize $Z = 8x_1 + 12x_2$
   subject to the constraints
   $$60x_1 + 30x_2 \ge 240$$
   $$30x_1 + 60x_2 \ge 300$$
   $$30x_1 + 180x_2 \ge 540$$
   $$x_1, x_2 \ge 0.$$

5. Minimize $Z = 3x + 2y$
   subject to the constraints
   $$5x + y \ge 10$$
   $$x + y \ge 6$$
   $$x + 4y \ge 12$$
   $$x, y \ge 0.$$

6. If x_1, x_2 be real, show that the set

$$S = \left\{ (x_1, x_2) : \begin{array}{|l} x_1 + x_2 \le 50 \\ x_1 + 2x_2 \le 80 \\ 2x_1 + x_2 \ge 20 \\ x_1, x_2 \ge 0 \end{array} \right\}$$

is a convex set. Find the extreme points of this set. Hence solve LPP graphically:
Maximize $Z = 4x_1 + 3x_2$ subject to the constraints given in S.

7. Minimize $Z = 20x + 10y$
subject to the constraints $x + 2y \le 40$
$3x + y \ge 30$
$4x + 3y \ge 60$
$x, y \ge 0$. (M.D.U. Dec. 2006)

8. (a) Maximize $Z = 2x_1 + 3x_2$
subject to the constraints $x_1 + x_2 \le 30$
$x_2 \ge 3$
$0 \le x_2 \le 12$
$0 \le x_1 \le 20$
$x_1 - x_2 \ge 0$
$x_1, x_2 \ge 0$. (M.D.U. Dec. 2005)

(b) Find a geometrical interpretation and solution as well for the following L.P. problem:
Maximize $Z = 3x_1 + 5x_2$
subject to restrictions $x_1 + 2x_2 \le 2000$
$x_1 + x_2 \le 1500$
$x_2 \le 600$
and $x_1 \ge 0, x_2 \ge 0$.

(c) Solve the following L.P.P. graphically:
Maximize $Z = 5x_1 + 3x_2$
subject to $4x_1 + 5x_2 \le 1000$
$5x_1 + 2x_2 \le 1000$
$3x_1 + 8x_2 \le 1200$
$x_1, x_2 \ge 0$ (K.U.K. Dec. 2010; M.D.U. May 2007)

(d) Using graphical method, solve the following L.P.P.
Maximize $Z = 2x_1 + 3x_2$ subject to
$x_1 - x_2 \le 2$
$x_1 + x_2 \ge 4$
$x_1, x_2 \ge 0$ (M.D.U. May 2011)

9. (a) A housewife wishes to mix two types of foods X and Y in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 11 units of vitamin B. Food X costs ₹ 60 per kg and food Y costs ₹ 80 per kg. Food X contains 3 units per kg of vitamin A and 5 units per kg of vitamin B while food Y contains 4 units per kg of vitamin A and 2 units per kg of vitamin B. Formulate the problem as an LPP to minimise the cost of mixture. (M.D.U. Dec. 2006)

(b) A house wife wishes to mix together two kinds of food, X and Y, in such a way that the mixture contains at least 10 units of vitamin A, 12 units of vitamin B and 8 units of vitamin C. The vitamin contents of one kg of food is given below:

	Vitamin A	Vitamin B	Vitamin C
Food X	1	2	3
Food Y	2	2	1

One kg of food X costs ₹ 6 and one kg of food Y costs ₹ 10. Formulate the above problem as an LPP to find the least cost of the mixture which will produce the diet and solve it.

(c) A manufacturer of furniture makes two products: chairs and tables. Processing of these products is done on two machines A and B. A chair requires 2 hours on machine A and 6 hours on machine B. A table requires 5 hours on machine A and no time on machine B. There are 16 hours of time per day available on machine A and 30 hours on machine B. Profit gained by the manufacturer from a chair is ₹ 10 and from a table is ₹ 50. What should be the daily production of the two? (M.D.U. May 2006, Dec. 2007)

10. A retired person wants to invest an amount of upto ₹ 20,000. His broker recommends investing in two types of bonds A and B, bond A yielding 10% return on the amount invested and bond B yielding 15% return on the amount invested. After some consideration, he decides to invest at least ₹ 5000 in bond A and no more than ₹ 8000 in bond B. He also wants to invest at least as much in bond A as in bond B. What should his broker suggest if he wants to maximize his return on investments? Formulate the above problem as an LPP and solve it.

11. A factory manufactures two types of screws, A and B, each type requiring the use of two machines, an automatic and a hand operated. It takes 4 minutes on the automatic and 6 minutes on hand operated machines to manufacture a package of screws A, while it takes 6 minutes on automatic and 3 minutes on the hand operated machines to manufacture a package of screws B. Each machine is available for at the most 4 hours on any day. The manufacturer can sell a package of screws A at a profit of 70 paise and screws B at a profit of ₹ 1. Assuming that he can sell all the screws he can manufacture, how many packages of each type should the factory owner produce in a day in order to maximize his profit? Determine the maximum profit.

12. A firm manufactures two products X and Y, each requiring the use of three machines M_1, M_2 and M_3. The time required for each product in hours and total time available in hours on each machine are as follows:

Machine	Product X	Product Y	Available time (in hours)
M_1	2	1	70
M_2	1	1	40
M_3	1	3	90

If the profit is ₹ 40 per unit for product X and ₹ 60 per unit for product Y, how many units of each product should be manufactured to maximize profit?

13. A dietician wishes to mix two types of food in such a way that the vitamin contents of the mixture contain at least 8 units of vitamin A and 10 units of vitamin C. Food I contains 2 units/kg of vitamin A and 1 unit/kg of vitamin C while food II contains 1 unit/kg of vitamin A and 2 units/kg of vitamin C. It costs ₹ 5 per kg to purchase food I and ₹ 7 per kg to purchase food II. Determine the minimum cost of such a mixture.

14. (a) An oil company has two depots, A and B, with capacities of 7000 l and 4000 l, respectively. The company is to supply oil to three petrol pumps D, E and F whose requirements are 4500 l, 3000 l and 3500 l, respectively. The distances (in km) between the depots and the petrol pumps are given in the following table:

To \ From	Distance (in km) A	B
D	7	3
E	6	4
F	3	2

Assuming that the transportation cost per km is ₹ 1 per l, how should the delivery be scheduled in order that the transportation cost in minimum?

(b) There are two factories located one at place P and the other at place Q. From these locations, a certain commodity is to be delivered to each of the three depots situated at A, B and C. The weekly requirements of the depots are respectively 5, 5 and 4 units of the commodity while the production capacity of the factories at P and Q are respectively 8 and 6 units. The cost of transportation per unit is given below:

From \ To	Cost (in ₹)		
	A	B	C
P	160	100	150
Q	100	120	100

How many units should be transported from each factory to each depot in order that the transportation cost is minimum? What will be the minimum transportation cost?

15. A manufacturer of patent medicines is preparing a production plan on medicines A and B. There are sufficient raw materials available to make 20,000 bottles of A and 40,000 bottles of B, but there are only 45,000 bottles into which either of the medicines can be put. Further, it takes 3 hours to prepare enough material to fill 1000 bottles of A and 1 hour to prepare enough material to fill 1000 bottles of B. There are 66 hours available for this operation. The profit is ₹ 8 per bottle for A and ₹ 7 per bottle for B. How should the manufacturer schedule his production in order to maximize his profit?

16. A small manufacturer has employed 5 skilled men and 10 semi-skilled men and makes an article in two qualities, a deluxe model and an ordinary model. The making of a deluxe model requires 2 hours of work by a skilled man and 2 hours of work by a semi-skilled man. The ordinary model requires 1 hour by a skilled man and 3 hours by a semi-skilled man. By union rule, no man may work for more than 8 hours per day. The manufacturer gains ₹ 15 on deluxe model and ₹ 10 on ordinary model. How many of each type should be made in order to maximize his total daily profit?

17. Minimize $Z = 3x_1 + 2x_2$

    subject to the constraints

$$5x_1 + x_2 \geq 10$$
$$x_1 + x_2 \geq 6$$
$$x_1 + 4x_2 \geq 12$$
$$x_1, x_2 \geq 0.$$

18. A firm plans to purchase at least 200 quintals of scrap containing high quality metal X and low quality metal Y. It decides that the scrap to be purchased must contain at least 100 quintals of X-metal and not more than 35 quintals of Y-metal. The firm can purchase the scrap from two suppliers (A and B) in unlimited quantities. The percentage of X and Y metals in terms of weight in the scrap supplied by A and B is given below:

Metals	Supplier A	Supplier B
X	25%	75%
Y	10%	20%

The price of A's scrap is ₹ 200 per quintal and that of B is ₹ 400 per quintal. The firm wants to determine the quantities that it should buy from the two suppliers so that total cost is minimized. Formulate this as an LPP and solve it.

19. G.J. Breweries Ltd. has two bottling plants, one located at 'G' and the other at 'J'. Each plant produces three drinks: whisky, beer and brandy, named A, B and C, respectively. The number of bottles produced per day are as follows:

Drink	Plant at	
	G	J
Whisky	1,500	1,500
Beer	3,000	1,000
Brandy	2,000	5,000

A market survey indicates that during the month of July, there will be a demand of 20,000 bottles of whisky, 40,000 bottles of beer and 44,000 bottles of brandy. The operating cost per day for plants at G and J are 600 and 400 monetary units. For how many days should each plant be run in July so as to minimize the production cost, while still meeting the market demand? Solve graphically.

20. Verify that the following problems have no feasible solution.

    (i) Maximize $Z = 15x + 20y$

    subject to the constraints

$$x + y \geq 12$$
$$6x + 9y \leq 54$$
$$15x + 10y \leq 90$$
$$x, y \geq 0$$

    (ii) Maximize $Z = 4x + 3y$

    subject to the constraints

$$x + y \leq 3$$
$$2x - y \leq 3$$
$$x \geq 4$$
$$x, y \geq 0$$

21. Verify that the following problems have multiple solutions.

    (i) Minimize $Z = 12x + 9y$

    subject to the constraints

$$3x + 6y \geq 36$$
$$4x + 3y \geq 24$$
$$x + y \leq 15$$
$$x, y \geq 0.$$

    (ii) Maximize $Z = 6x + 4y$

    subject to the constraints

$$x + y \leq 5$$
$$3x + 2y \leq 12$$
$$x, y \geq 0$$

22. Verify that the following problem has an unbounded solution:

    Maximize $Z = 20x + 30y$

    subject to the constraints

$$5x + 2y \geq 50$$
$$2x + 6y \geq 20$$
$$4x + 3y \geq 60$$
$$x, y \geq 0$$

23. Graph the feasible region of the following problem and identify a redundant constraint.

    Minimize $Z = 6x + 10y$

    subject of the constraints

$$x \geq 6$$
$$y \geq 2$$
$$2x + y \geq 10$$
$$x, y \geq 0.$$

Answers

1. $x_1 = \frac{8}{5}, x_2 = \frac{12}{5}$; Max. $Z = 24.8$

2. $x = 18,\ y = 12$; Max. $Z = 72$

3. $x_1 = \frac{20}{19}, x_2 = \frac{45}{19}$; Max. $Z = \frac{235}{19}$

4. $x_1 = 2, x_2 = 4$; Min. $Z = 64$

5. $x_1 = 1, x_2 = 5$; Min. $Z = 13$

6. $x_1 = 50, x_2 = 0$; Max. $Z = 200$

7. $x = 6, y = 12$; Min. $Z = 240$

8. (a) $x_1 = 18, x_2 = 12$; Max. $Z = 72$ (b) $x_1 = 1000, x_2 = 500$; Max. $Z = 5500$

   (c) $x_1 = \frac{3000}{17}, x_2 = \frac{1000}{17}$, Max. $Z = 1058.82$ (d) No maximum

9. (a) Minimum cost = ₹ 160
(b) 2 kg of food X, 4 kg of food Y; least cost = ₹ 52
(c) Number of chairs = 0, Number of tables = 16/5 *i.e.*, 16 in 5 days
10. ₹ 12000 in bond A, ₹ 8,000 in bond B; Maximum return = ₹ 2400
11. Screw A = 30 packages, Screw B = 20 packages; Maximum profit = ₹ 41
12. Product X = 15, Product Y = 25; Maximum profit = ₹ 2100
13. Food I = 2 kg, Food II = 4 kg; Minimum cost = ₹ 38
14. (a) From A: 500 *l*, 3000 *l*, 3500 *l* to D, E, F respectively, From B: 4000 *l*, 0 *l*, 0 *l* to D, E, F respectively; Minimum cost = ₹ 44,000
(b) From P: 0, 5, 3 units to A, B, C respectively; From Q: 5, 0, 1 units to A, B, C respectively, Minimum transportation cost = ₹ 1550.
15. 10500 bottles of A, 34500 bottles of B; Maximum profit = ₹ 325500
16. Delux model = 10, ordinary model = 20; Maximum profit = ₹ 350
[**Hint.** 5 skilled men cannot work for more than 5 × 8 = 40 hours and 10 semi-skilled men cannot work for more than 10 × 8 = 80 hours.]
17. $x_1 = 1, x_2 = 5$; Min. Z = 13
18. Supplier A = 100 quintals, supplier B = 100 quintals; Minimum cost = ₹ 60,000
19. Plant at G = 12 days, Plant at J = 4 days; Minimum cost = 8,800 monetary units.
23. $2x + y \geq 10$.

THE SIMPLEX METHOD

7.7. INTRODUCTION

The linear programming problems discussed so far are concerned with two variables, the solution of which can be found out easily by the graphical method. But most real life problems when formulated as an LP model involve more than two variables and many constraints. Thus, there is a need for a method other than the graphical method. The most popular non-graphical method of solving an LPP is called the **simplex method.** This method, developed by George B. Dantzig in 1947, is applicable to any problem that can be formulated in terms of linear objective function subject to a set of linear constraints. There are no theoretical restrictions placed on the number of decision variables or constraints. The development of computers has further made it easy for the simplex method to solve large scale LP problems very quickly.

The concept of simplex method is similar to the graphical method. For LP problems with several variables, the optimal solution lies at a corner point of the many-faced, multi-dimensional figure, called an n-dimensional polyhedron. The simplex method examines the corner points in a systematic manner. It is a computational routine of repeating the same set of steps over and over until an optimal solution is reached. For this reason, it is known as an **iterative method.** As we move from one iteration to the other, the method improves the value of the objective function and achieves optimal solution in a finite number of iterations.

7.8. SOME USEFUL DEFINITIONS

(*i*) **Slack Variable.** A variable added to the left hand side of a less than or equal to constraint to convert the constraint into an equality is called a *slack variable.*

For example, to convert the constraint

$$3x + 2y \leq 18 \qquad ...(1)$$

into an equation, we add a slack variable s to the left-hand side thereby getting the equality

$$3x + 2y + s = 18$$

Clearly, s must be non-negative, since $s = 18 - (3x + 2y) \geq 0$ [by (1)]

In economic terminology, a slack variable represents unused resource in the form of money, labour hours, time on a machine etc.

(*ii*) **Surplus Variable.** A variable subtracted from the left-hand side of a greater than or equal to constraint to convert the constraint into an equality is called a *surplus variable.*

For example, to convert the constraint

$$2x + 3y \geq 30 \qquad ...(1)$$

into an equation, we subtract a surplus variable s from the left-hand side thereby getting the equality

$$2x + 3y - s = 30$$

Clearly, s must be non-negative, since $s = (2x + 3y) - 30 \geq 0$ [by (1)]

A surplus variable represents the surplus of left-hand side over the right-hand side. It is also called a negative slack variable.

(*iii*) **Basic Solution.** For a system of m simultaneous linear equations in n variables $(n > m)$, a solution obtained by setting $(n - m)$ variables equal to zero and solving for the remaining m variables for a unique solution is called a **basic solution.** The $(n - m)$ variables set equal to zero in any solution are called **non-basic variables.** The other m variables are called **basic variables.**

Total number of basic solution = ${}^nC_m$

(*iv*) **Basic Feasible Solution.** A basic solution which happens to be feasible (*i.e.*, a solution in which each basic variable is non-negative) is called a basic feasible solution.

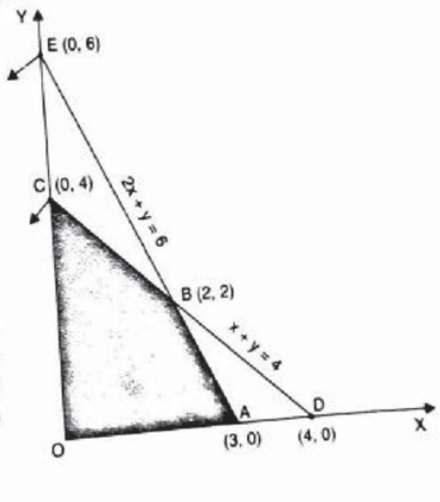
(*v*) **Degenerate and Non-degenerate Solution.** If one or more of the basic variables in the basic feasible solution are zero, then it is called a **degenerate solution.** If all the variables in the basic feasible solution are positive, then it is called a non-degenerate solution.

Example. *Consider the following LPP:*

$$\text{Maximize } Z = 5x + 8y$$

subject to the constraints

$$x + y \leq 4$$
$$2x + y \leq 6$$
$$x, y \geq 0.$$



Sol. Introducing slack variables s_1, s_2 to convert less than or equal to constraints into equalities, we get

$$x + y + s_1 = 4$$
$$2x + y + s_2 = 6$$

We have two equations (m = 2) in four variables (n = 4). For a basic solution, we put $n - m = 4 - 2 = 2$ variables equal to zero and solve for the other two. The various possibilities are shown in the following table:

Number of basic solution	*Non-basic variables (each = 0)*	*Basic variables*	*Basic solutions*
1	s_1, s_2	x, y	$x = 2, y = 2$
2	x, s_1	y, s_2	$y = 4, s_2 = 2$
3	x, s_2	y, s_1	$y = 6, s_1 = -2$
4	y, s_1	x, s_2	$x = 4, s_2 = -2$
5	y, s_2	x, s_1	$x = 3, s_1 = 1$
6	x, y	s_1, s_2	$s_1 = 4, s_2 = 6$

There are ${}^nC_m = {}^4C_2 = 6$ basic solutions. Solutions (3) and (4) contain a variable which has a negative value. In the remaining four basic solutions, each variable has a positive value. Therefore solutions (1), (2), (5) and (6) are basic feasible solutions. The four basic feasible solutions (x, y) are:

$$(0, 0), \quad (3, 0), \quad (2, 2) \quad \text{and} \quad (0, 4)$$

which correspond to the corner points O, A, B and C respectively of the feasible region as shown in the figure.

Thus every corner point of the feasible region corresponds to a basic feasible solution and *vice versa.*

Moreover, the basic feasible solutions (0, 0), (3, 0) and (0, 4) are degenerate whereas the basic feasible solution (2, 2) is non-degenerate.

7.9. STANDARD FORM OF AN LPP

The standard form of an LPP should have the following characteristics:

(*i*) Objective function should be of maximization type.

(*ii*) All constraints should be expressed as equations by adding slack or surplus variables, one for each constraint.

(*iii*) The right-hand side of each constraint should be non-negative. If it is negative, then to make it positive, we multiply both sides of the constraint by (– 1), changing ≤ to ≥ and *vice versa.*

(*iv*) All variables are non-negative.

Thus, the standard form of an LPP with n variables and m constraints is:

Maximize $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n + 0s_1 + 0s_2 + \ldots\ldots + 0s_m$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n + s_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n + s_2 = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n + s_m = b_m$$

and $\quad x_1, x_2 \ldots\ldots, x_n, s_1, s_2, \ldots\ldots, s_m \geq 0$

Remarks 1. Since minimum Z = – maximum Z*, where Z* = – Z, the objective function can always be expressed in the maximization form.

2. Replace each unrestricted variable x_i with the difference of two non-negative variables. Thus $x_i = x_i' - x_i''$, where $x_i', x_i'' \geq 0$.

ILLUSTRATIVE EXAMPLES

Example 1. *Express the following LPP in the standard form:*

Maximize $Z = 2x_1 + 5x_2 + 7x_3$

subject to the constraints $\quad 5x_1 - 3x_2 \leq 4$

$$7x_1 + 6x_2 + 9x_3 \geq 15$$
$$8x_1 + 6x_3 \leq 5$$
$$x_1, x_2, x_3 \geq 0$$

Sol. Introducing the slack/surplus variables, the given LPP in standard form becomes:

Maximize $Z = 2x_1 + 5x_2 + 7x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$5x_1 - 3x_2 + s_1 = 4$$
$$7x_1 + 6x_2 + 9x_3 - s_2 = 15$$
$$8x_1 + 6x_3 + s_3 = 5$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Example 2. *Convert the following LPP to the standard form:*

Maximize $Z = 2x_1 + 4x_2 + 8x_3$

subject to the constraints $\quad 7x_1 - 4x_2 \leq 6$

$$4x_1 + 3x_2 + 6x_3 \geq 15$$
$$3x_1 + 2x_3 \leq 4$$
$$x_1, x_2 \geq 0.$$

Sol. Here x_3 is unrestricted, so let $x_3 = x_3' - x_3''$, where $x_3', x_3'' \geq 0$

Introducing the slack/surplus variables, the given LPP in standard form becomes:

Maximize $Z = 2x_1 + 4x_2 + 8x_3' - 8x_3'' + 0s_1 + 0s_2 + 0s_3$

subject to the constraints. $\quad 7x_1 - 4x_2 + s_1 = 6$

$$4x_1 + 3x_2 + 6x_3' - 6x_3'' - s_2 = 15$$
$$3x_1 + 2x_3' - 2x_3'' + s_3 = 4$$
$$x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0.$$

Example 3. *Express the following LPP in the standard form:*

Minimize $Z = 5x_1 + 8x_2$

subject to the constraints $\quad 3x_1 + 7x_2 + x_3 = 9$

$$4x_1 - 2x_2 - x_4 = -15$$
$$2x_1 - 3x_2 + x_5 = 8$$
$$x_1, x_3, x_4, x_5 \geq 0.$$

Sol. Here x_1, x_2 are the only two decision variables and x_3, x_4, x_5 are the slack/surplus variables. Also x_2 is unrestricted, so let $x_2 = x_2' - x_2''$, where $x_2', x_2'' \geq 0$. Since right hand side of second constraint is negative, we multiply throughout by (-1).

$\therefore$ The given problem in standard form is:

Maximize $Z^* (= -Z) = -5x_1 - 8x_2' + 8x_2'' + 0x_3 + 0x_4 + 0x_5$

subject to the constraints
$$3x_1 + 7x_2' - 7x_2'' + x_3 = 9$$
$$-4x_1 + 2x_2' - 2x_2'' + x_4 = 15$$
$$2x_1 - 3x_2' + 3x_2'' + x_5 = 8$$
$$x_1, x_2', x_2'', x_3, x_4, x_5 \geq 0.$$

[Min. Z = – Max. Z*]

EXERCISE 7.4

1. Find all the basic solutions of the following system of equations identifying in each case the basic and non-basic variables:
$$2x_1 + x_2 + 4x_3 = 11, \quad 3x_1 + x_2 + 5x_3 = 14.$$
Investigate whether the basic solutions are degenerate basic solutions or not. Hence find the basic feasible solution of the system.

2. Obtain all the basic solutions to the following system of linear equations:
$$x_1 + 2x_2 + x_3 = 4, \quad 2x_1 + x_2 + 5x_3 = 5$$
Which of them are feasible? Point out non-degenerate basic solutions, if any.

3. Show that the following system of linear equations has two degenerate feasible solutions and the non-degenerate basic solution is not feasible.
$$2x_1 + x_2 - x_3 = 2, \quad 3x_1 + 2x_2 + x_3 = 3.$$

4. Find all the basic solutions to the following problem:

Maximize $Z = x_1 + 3x_2 + 3x_3$

subject to the constraints
$$x_1 + 2x_2 + 3x_3 = 4$$
$$2x_1 + 3x_2 + 5x_3 = 7$$
$$x_1, x_2, x_3 \geq 0$$
Which of the basic solutions are

(*i*) non-degenerate basic feasible (*ii*) optimal basic feasible?

5. Find an optimal solution to the following LPP by computing all basic solutions and then finding one that maximizes the objective function:

Maximize $Z = 2x_1 + 3x_2 + 4x_3 + 7x_4$

subject to the constraints
$$2x_1 + 3x_2 - x_3 + 4x_4 = 8$$
$$x_1 - 2x_2 + 6x_3 - 7x_4 = -3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

6. Express the following LP problems in the standard form:

(*i*) Maximize $Z = 3x_1 + 4x_2 + 5x_3$

subject to the constraints
$$2x_1 + 7x_2 + x_3 \leq 10$$
$$5x_1 + 9x_2 + 4x_3 \geq 20$$
$$8x_1 + 15x_3 \leq 30$$
$$x_1, x_2, x_3 \geq 0$$

(*ii*) Maximize $Z = x_1 + 7x_2 + 2x_3$

subject to the constraints
$$3x_1 + 2x_2 + x_3 \leq 8$$
$$5x_1 + 7x_2 \geq 14$$
$$4x_2 + 3x_3 \leq 12$$
$$x_1, x_2 \geq 0$$

(*iii*) Minimize $Z = x_1 - 3x_2 + 3x_3$

subject to the constraints
$$3x_1 - x_2 + 2x_3 \leq 7$$
$$2x_1 + 4x_2 \geq -12$$
$$-4x_1 + 3x_2 + 8x_3 \leq 10$$
$$x_1, x_2, x_3 \geq 0$$

Answers

1.

Non-basic variables	*Basic variables*	*Basic solutions*
x_3	x_1, x_2	$x_1 = 3,\ x_2 = 5$
x_1	x_2, x_3	$x_2 = -1,\ x_3 = 3$
x_2	x_1, x_3	$x_1 = \frac{1}{2},\ x_3 = \frac{5}{2}$

First and third are basic feasible solutions which are also non-degenerate basic solutions.

2. $x_1 = 2,\ x_2 = 1$; feasible, non-degenerate

$x_2 = \dfrac{5}{3},\ x_3 = \dfrac{2}{3}$; feasible, non-degenerate

$x_1 = 5,\ x_3 = -1$; non-feasible.

4. $x_1 = 2, x_2 = 1$; $(x_3 = 0)$ $\quad$ $x_1 = x_3 = 1$; $(x_2 = 0)$ $\quad$ $x_2 = -1, x_3 = 2$ $(x_1 = 0)$

(*i*) First two solutions are non-degenerate basic feasible solutions

(*ii*) First solution is optimal basic feasible and Max. Z = 5.

5. Optimal basic feasible solution is $x_1 = 0, x_2 = 0, x_3 = \dfrac{44}{17}, x_4 = \dfrac{45}{17}$ and Max. $Z = \dfrac{491}{17}$

6. (*i*) Maximize $Z = 3x_1 + 4x_2 + 5x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints
$$2x_1 + 7x_2 + x_3 + s_1 = 10$$
$$5x_1 + 9x_2 + 4x_3 - s_2 = 20$$
$$8x_1 + 15x_3 + s_3 = 30$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

(*ii*) Maximize $Z = x_1 + 7x_2 + 2x_3' - 2x_3'' + 0s_1 + 0s_2 + 0s_3$

subject to the constraints
$$3x_1 + 2x_2 + x_3' - x_3'' + s_1 = 8$$
$$5x_1 + 7x_2 - s_2 = 14$$
$$4x_2 + 3x_3' - 3x_3'' + s_3 = 12$$
$$x_1, x_2, x_3', x_3'', s_1, s_2, s_3 \geq 0$$

(*iii*) Maximize $Z^*(= -Z) = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$

subject to the constraints

$$3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 - 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

7.10. WORKING PROCEDURE OF SIMPLEX METHOD

Assuming the existence of an initial basic feasible solution, the optimal solution to an LPP by simplex method is obtained as follows:

Step 1. To express the LPP in the Standard Form

(*i*) Formulate the mathematical model of the given LPP.

(*ii*) If the objective function is to be minimized, then convert it into a maximization problem by using

$$\text{Min. } Z = -\text{Max. }(-Z)$$

(*iii*) The right-hand side of each constraint should be non-negative.

(*iv*) Express all constraints as equations by introducing slack/surplus variables, one for each constraint.

(*v*) Restate the given LPP in standard form:

Maximize $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n + 0s_1 + 0s_2 + \ldots\ldots + 0s_m$

subject to the constraints $\quad a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n + s_1 = b_1$

$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n + s_2 = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n + s_m = b_m$$

and $\quad x_1, x_2, \ldots\ldots, x_n, s_1, s_2, \ldots\ldots, s_m \geq 0$

Step 2: To set up the Initial Basic Feasible Solution

Take the m slack/surplus variables $s_1, s_2, \ldots\ldots, s_m$ as the basic variables so that the n given variables $x_1, x_2, \ldots\ldots, x_n$ are non-basic variables. As such, $x_1 = x_2 = \ldots\ldots = x_n = 0$ and $s_1 = b_1, s_2 = b_2, \ldots\ldots, s_m = b_m$.

Since each b_i is non-negative [see Step I (*iii*)], the basic solution is feasible. This basic feasible solution is the starting point of the iterative process. The simplex method then proceeds to solve the LPP by designing and re-designing successively better basic feasible solutions until an optimal solution is obtained.

Step 3: To set up the Initial Simplex Table

The above information is conveniently expressed in the tabular form as shown below. For computational efficiency, the tabular form is better.

		$c_j \to$	c_1	c_2	$\ldots\ldots c_n$	0	0	$\ldots 0$
C_B	*Basic Variables* B	*Solution* $b(= x_B)$	*Variables* x_1	x_2	$\ldots\ldots x_n$	s_1	s_2	$\ldots\ldots s_m$
$(C_{B1} =)\ 0$	s_1	$(x_{B1} =) b_1$	a_{11}	a_{12}	$\ldots\ldots a_{1n}$	1	0	$\ldots\ldots 0$
$(C_{B2} =)\ 0$	s_2	$(x_{B2} =) b_2$	a_{21}	a_{22}	$\ldots\ldots a_{2n}$	0	1	$\ldots\ldots 0$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
$(C_{Bm} =)\ 0$	s_m	$(x_{Bm} =) b_m$	a_{m1}	a_{m2}	$\ldots\ldots a_{mn}$	0	0	$\ldots\ldots 1$
$Z = \Sigma C_{Bi}\, x_{Bi}$	$Z_j = \Sigma C_{Bi} a_{ij}$		0	0	$\ldots\ldots 0$	0	0	$\ldots\ldots 0$
	$C_j = c_j - Z_j$		$c_1 - Z_1$	$c_1 - Z_2$	$\ldots\ldots c_n - Z_n$	0	0	$\ldots\ldots 0$

The data in the table is interpreted as follows:

1. In the top row labelled c_j, we write the co-efficients of the variables in the objective function. These values will remain the same in subsequent simplex tables.
2. The second row shows the major column headings for the simplex table. These headings remain the same in subsequent simplex tables and apply to the values listed in the first m-rows.
3. In the first column labelled 'c_B', we write the co-efficients of the current basic variables in the objective function. c_{Bi} is the co-efficient of the i^{th} basic variable.
4. In the second column labelled **'Basic Variables'**, we write the basic variables. This column can also be labelled as **'Basis'**.
5. In the column labelled **'Solution'**, we write the current values of the corresponding basic variables. The remaining non-basic variables will always be zero.
6. The **body matrix** under non-basic variables $x_1, x_2, \ldots\ldots, x_n$ consists of the co-efficients of the decision variables in the constraint set.
7. The **identity matrix** under $s_1, s_2, \ldots\ldots, s_m$ represents the co-efficients of the slack variables in the constraint set.
8. To get an entry in the z_j-row under a column, we multiply the entries in that column by the corresponding entries of c_B column and add up the products. Thus the entry in the z_j-row under the column 'x_2' is obtained by multiplying the entries under 'x_2', *viz.*, $a_{12}, a_{22}, \ldots\ldots, a_{m2}$ by the corresponding entries under c_B, *viz.*, $c_{B1}, c_{B2}, \ldots\ldots, c_{Bm}$ and adding, thereby getting

$$c_{B1}a_{11} + c_{B2}a_{22} + \ldots\ldots + c_{Bm}a_{m2}$$

In the initial simplex table, the z_j-row entries will be all equal to zero. The Z entry under the C_B column gives the current value of the objective function. Thus $Z = \Sigma c_{Bi}x_{Bi} = c_{B1}x_{B1} + c_{B2}x_{B2} + \ldots\ldots + c_{Bm}x_{Bm}$. The values of z_j represent the amount by which the value of objective function Z would be decreased (or increased) if one of the variables not included in the solution were brought into the solution.

9. The last row labelled '$C_j = c_j - Z_j$', is called the **index row** or **net evaluation row.**

C_j for any column = (c_j value for that column written at the top of that column) − (Z_j value for that column)

It should be noted that C_j values are meaningful for non-basic variables only. For a basic variable, $C_j = 0$.

The C_j-row represents the net contribution to the objective function that results by introducing one unit of each of the respective column variables. A plus value indicates that a greater contribution can be made by bringing the variable for that column into the solution. A negative value indicates the amount by which contribution would decrease if one unit of the variable for that column were brought into the solution.

Step 4: Test for Optimality. Examine the entries in C_j-row. If all entries in this row are negative or zero, *i.e.*, if $C_j \leq 0$, then the basic feasible solution is optimal. Any positive entry in the row indicates that an improvement in the value of objective function Z is possible and, hence, we proceed to the next step.

Step 5: To Identify the Incoming and Outgoing Variables. If there is a positive entry in the C_j-row, then simplex method shifts from the current basic feasible solution to a better basic feasible solution. For this, we have to replace one current basic variable (called the **outgoing or departing variable**) by a new non-basic variable (called the **incoming or entering variable**).

Determination of Incoming Variable. The column with the largest positive entry in the C_j-row is called the **key** (or **pivot**) **column** (which is shown marked with an arrow ↑). The non-basic variable which will replace a basic variable is the one lying in the key column. Thus the incoming variable is located.

If more than one variable has the same positive largest entry in the C_j-row, then any of these variables may be selected arbitrarily as the incoming variable.

Determination of Outgoing Variable. Divide each entry of the solution column (*i.e.*, x_B-column) by the corresponding positive entry in the key column. These quotients are written in the last column labelled 'Ratio'. The row which corresponds to the smallest non-negative quotient is called the **key** (or **pivot**) **row** (which is shown marked with an arrow →). The departing variable is the corresponding basic variable in this row. The element at the intersection of the key row and key column is called the **key** (or **pivot**) **element.** We place a circle around this element.

If all these ratios are negative or zero, the incoming variable can be made as large as we please without violating the feasibility condition. Hence the problem has an unbounded solution and no further iteration is required.

Step 6: To set up the new simplex table from the current one.

Drop the outgoing variable and introduce the incoming variable alongwith its associated value under c_B column.

If the key element is 1, then the key row remains the same in the new simplex table.

If the key element is not 1, then to reduce it to 1, divide each element in the key row (including elements in x_B-column) by the key element.

Thus, $\quad$ new key row = $\dfrac{\text{old key row}}{\text{key element}}$

Make all other elements of the key column 0 by subtracting suitable multiples of key row from the other rows. In other words, to change the non-key rows, we use the formula:

Number in new non-key row = (Number in old non-key row) − (key column entry) × (corresponding number in new key-row)

where **'key column entry'** is the entry in this row that is in the key column.

Step 7: Test for Optimality. If there is no positive entry in C_j-row, we have an optimal solution. Otherwise, go to Step 4 and repeat the procedure until all entries in C_j-row are either negative or zero.

ILLUSTRATIVE EXAMPLES

Example 1. *Using simplex method*

Maximize $Z = 2x_1 + 5x_2$

subject to
$$x_1 + 4x_2 \leq 24$$
$$3x_1 + x_2 \leq 21$$
$$x_1 + x_2 \leq 9$$
$$x_1, x_2 \geq 0.$$
(M.D.U. Dec. 2007)

Sol. Step 1. To formulate the mathematical model of the problem. We are already given the mathematical model of the LPP. The problem is of maximization type and all b's are positive.

Step 2. To express the problem in standard form. Introducing slack variables s_1, s_2, s_3 (one for each constraint) the problem in standard form is:

Maximize $\quad Z = 2x_1 + 5x_2 + 0s_1 + 0s_2 + 0s_3$

subject to
$$x_1 + 4x_2 + s_1 = 24 \quad \ldots(1)$$
$$3x_1 + x_2 + s_2 = 21 \quad \ldots(2)$$
$$x_1 + x_2 + s_3 = 9 \quad \ldots(3)$$
$$x_1, x_2, s_1, s_2, s_3 \geq 0$$

Step 3. To set up the initial basic feasible solution. Since we have 3 equations in 5 variables, a solution is obtained by setting $5 - 3 = 2$ variables equal to zero and solving for the remaining 3 variables. We start with a basic solution by setting $x_1 = x_2 = 0$. Substituting $x_1 = x_2 = 0$ in (1), (2) and (3), we get the basic solution
$$s_1 = 24, \quad s_2 = 21, \quad s_3 = 9$$

Since s_1, s_2, s_3 are all positive, the basic solution is feasible and non-degenerate.

Thus our initial basic feasible solution is
$$x_1 = 0, \quad x_2 = 0; \quad s_1 = 24, \quad s_2 = 21, \quad s_3 = 9$$
at which $Z = 0$. This initial basic feasible solution is summarized in the following initial simplex table.

Simplex Table I

c_B	Basis	$c_j \to$	2	5	0	0	0	Ratio
		Solution $b(=x_B)$	x_1	x_2	s_1	s_2	s_3	x_B/x_2
0	s_1	24	1	4	1	0	0	$\frac{24}{4} = 6 \to$
0	s_2	21	3	1	0	1	0	$\frac{21}{1} = 21$
0	s_3	9	1	1	0	0	1	$\frac{9}{1} = 9$
Z = 0		Z_j	0	0	0	0	0	
		$C_j = c_j - Z_j$	2	5 ↑	0	0	0	

*For writing the body matrix (under x_1, x_2) and the identity matrix (under s_1, s_2, s_3), the left hand sides of equations (1), (2), (3) should be treated as

$$1x_1 + 4x_2 + 1s_1 + 0s_2 + 0s_3$$
$$3x_1 + 1x_2 + 0s_1 + 1s_2 + 0s_3$$
$$1x_1 + 1x_2 + 0s_1 + 0s_2 + 1s_3$$

*The Z_j-row entries are all equal to zero in the initial simplex table.

Step 4. To test for optimality. Since some entries in C_j-row are positive, the current solution is not optimal. Therefore, an improvement in the value of objective function Z is possible and we proceed to the next step.

Step 5. To identify the incoming and outgoing variables. The largest positive entry in the C_j-row is 5 and the column in which it appears is headed by x_2. Thus x_2 is the incoming variable and x_2-column is the key column (indicated by ↑).

Dividing each entry of the solution column by the corresponding positive entry in the key column, we find that minimum positive ratio is 6 and it appears in the row headed by s_1. Thus s_1 is the outgoing basic variable and the corresponding row is the key row (indicated by →). The number at the intersection of the key row and the key column is the key number. Thus 4 is the key number. A circle is placed around this number.

Step 6. To set up the new simplex table from the current one. Drop the outgoing variable s_1 from the basis and introduce the incoming variable x_2. The new basis will contain x_2, s_2, s_3 as the basic variables. The co-efficient of x_2 in the objective function is 5. Therefore the entry in c_B column corresponding to the new basic variable x_2 will be 5. Since the key element enclosed in the circle is not 1, divide all elements of the key row by the key element 4 to obtain new values of the elements in this row. Thus the key row

$$0 \quad s_1 \quad 24 \quad 1 \quad 4 \quad 1 \quad 0 \quad 0$$

is replaced by the **new key row**

$$5 \quad x_2 \quad 6 \quad \frac{1}{4} \quad 1 \quad \frac{1}{4} \quad 0 \quad 0$$

Now we make all other elements of the key column (*i.e.*, x_2-column) zero by subtracting suitable multiples of key row from the other rows.

To replace non-key rows, we use the formula:

Number in new non-key row = (Number in old non-key row) – (key column entry) × (corresponding number in new key row)

Here s_2-row and s_3-row (*i.e.*, second and third rows) are non-key rows.

Transformation of R_2. Key column entry in R_2 is 1.

$$\therefore \quad R_2(\text{new}) = R_2(\text{old}) - 1 \times R_1(\text{new})$$

$$21 - 1 \times 6 = 15$$
$$3 - 1 \times \frac{1}{4} = \frac{11}{4}$$
$$1 - 1 \times 1 = 0$$
$$0 - 1 \times \frac{1}{4} = -\frac{1}{4}$$
$$1 - 1 \times 0 = 1$$
$$0 - 1 \times 0 = 0$$

Transformation of R_3. Key column entry in R_3 is 1.

$$\therefore \quad R_3(\text{new}) = R_3(\text{old}) - 1 \times R_1(\text{new})$$

$$9 - 1 \times 6 = 3$$
$$1 - 1 \times \frac{1}{4} = \frac{3}{4}$$
$$1 - 1 \times 1 = 0$$
$$0 - 1 \times \frac{1}{4} = -\frac{1}{4}$$
$$0 - 1 \times 0 = 0$$
$$1 - 1 \times 0 = 1$$

The above information is summarized in the following table:

Simplex Table II

c_B	Basis	$c_j \to$	2	5	0	0	0	Ratio
		Solution $b(=x_B)$	x_1	x_2	s_1	s_2	s_3	x_B/x_1
5	x_2	6	$\frac{1}{4}$	1	$\frac{1}{4}$	0	0	$\frac{6}{(1/4)} = 24$
0	s_2	15	$\frac{11}{4}$	0	$-\frac{1}{4}$	1	0	$\frac{15}{(11/4)} = \frac{60}{11}$
0	s_3	3	$\frac{3}{4}$	0	$-\frac{1}{4}$	0	1	$\frac{3}{(3/4)} = 4 \to$
Z = 30		Z_j	$\frac{5}{4}$	5	$\frac{5}{4}$	0	0	
		$C_j = c_j - Z_j$	$\frac{3}{4}$ ↑	0	$-\frac{5}{4}$	0	0	

$*Z = \Sigma c_{Bi} x_{Bi} = 5(6) + 0(15) + 0(3) = 30$

* **Entries in Z_j-row.** In the column headed by

x_1	$5\left(\frac{1}{4}\right)+0\left(\frac{11}{4}\right)+0\left(\frac{3}{4}\right)$	$=\frac{5}{4}$
x_2	$5(1)+0(0)+0(0)$	$=5$
s_1	$5\left(\frac{1}{4}\right)+0\left(-\frac{1}{4}\right)+0\left(-\frac{1}{4}\right)$	$=\frac{5}{4}$
s_2	$5(0)+0(1)+0(0)$	$=0$
s_3	$5(0)+0(0)+0(1)$	$=0$

Step 7. Test for optimality. Since all entries in C_j-row are not negative or zero, the current solution is not optimal. Therefore, an improvement in the value of objective function Z is possible and we repeat steps 5 to 7.

Incoming variable is x_1 and x_1-column is the key-column. s_3 is the outgoing basic variable and s_3-row is the key row. Key number is $\frac{3}{4}$.

The new basis will contain x_2, s_2, x_1 as the basic variables. Dividing all elements of the key row by the key number $\frac{3}{4}$, the new key row is

$$2 \quad x_1 \quad 4 \quad 1 \quad 0 \quad -\frac{1}{3} \quad 0 \quad \frac{4}{3}$$

Transformation of R_1. Key column entry in R_1 is $\frac{1}{4}$

$$\therefore \quad R_1(\text{new}) = R_1(\text{old}) - \frac{1}{4}R_3(\text{new})$$

$$6-\frac{1}{4}(4)=5$$
$$\frac{1}{4}-\frac{1}{4}(1)=0$$
$$1-\frac{1}{4}(0)=1$$
$$\frac{1}{4}-\frac{1}{4}\left(-\frac{1}{3}\right)=\frac{1}{3}$$
$$0-\frac{1}{4}(0)=0$$
$$0-\frac{1}{4}\left(\frac{4}{3}\right)=-\frac{1}{3}$$

Transformation of R_2. Key column entry in R_2 is $\frac{11}{4}$.

$$\therefore \quad R_2(\text{new}) = R_2(\text{old}) - \frac{11}{4}R_3(\text{new})$$

$$15-\frac{11}{4}(4)=4$$
$$\frac{11}{4}-\frac{11}{4}(1)=0$$
$$0-\frac{11}{4}(0)=0$$

$$-\frac{1}{4}-\frac{11}{4}\left(-\frac{1}{3}\right)=\frac{2}{3}$$
$$1-\frac{11}{4}(0)=1$$
$$0-\frac{11}{4}\left(\frac{4}{3}\right)=-\frac{11}{3}$$

The above information is summarized in the following table.

Simplex Table III

c_B	Basis	$c_j \rightarrow$ Solution $b(=x_B)$	2 x_1	5 x_2	0 s_1	0 s_2	0 s_3	Ratio
5	x_2	5	0	1	$\frac{1}{3}$	0	$-\frac{1}{3}$	
0	s_2	4	0	0	$\frac{2}{3}$	1	$-\frac{11}{3}$	
2	x_1	4	1	0	$-\frac{1}{3}$	0	$\frac{4}{3}$	
Z = 33		Z_j	2	5	1	0	1	
		$C_j = c_j - Z_j$	0	0	−1	0	−1	

* $Z = \Sigma\, c_{Bi}x_{Bi} = 5(5) + 0(4) + 2(4) = 33$

As all entries in C_j-row are either negative or zero, the above table gives the optimal basic feasible solution.

$\therefore$ The optimal solution is $x_1 = 4$, $x_2 = 5$ and Max. Z = 33.

Example 2. *Use the simplex method to solve the following LP problem.*

Maximize $\quad Z = 3x_1 + 5x_2 + 4x_3$

subject to

$$2x_1 + 3x_2 \le 8$$
$$2x_2 + 5x_3 \le 10$$
$$3x_1 + 2x_2 + 4x_3 \le 15$$
$$x_1, x_2, x_3 \ge 0$$

Sol. Step 1. Introducing slack variables s_1, s_2, s_3 (one for each constraint) the problem in standard form is:

Maximize $\quad Z = 3x_1 + 5x_2 + 4x_3 + 0s_1 + 0s_2 + 0s_3$ ...(1)

subject to

$$2x_1 + 3x_2 + s_1 = 8 \quad \text{...(2)}$$
$$2x_2 + 5x_3 + s_2 = 10 \quad \text{...(3)}$$
$$3x_1 + 2x_2 + 4x_3 + s_3 = 15$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \ge 0$$

Step 2. Since we have 3 equations in 6 variables, a solution is obtained by setting 6 − 3 = 3 variables equal to zero and solving for the remaining 3 variables. We choose initial basic feasible solution as:

$$x_1 = x_2 = x_3 = 0; \quad s_1 = 8, \quad s_2 = 10, \quad s_3 = 15$$

at which Z = 0.

Simplex Table I

c_B	Basis	$c_j \rightarrow$ Solution $b(=x_B)$	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Ratio x_B/x_2
0	s_1	8	2	3	0	1	0	0	$\frac{8}{3} \rightarrow$
0	s_2	10	0	2	5	0	1	0	$\frac{10}{2} = 5$
0	s_3	15	3	2	4	0	0	1	$\frac{15}{2}$
$Z = 0$		Z_j	0	0	0	0	0	0	
		$C_j = c_j - Z_j$	3	5 $\uparrow$	4	0	0	0	

Step 3. Since some entries in C_j-row are positive, the current solution is not optimal. The incoming variable is x_2 and the outgoing basic variable is s_1. Also 3 is the key number.

Step 4. The co-efficient of x_2 in the objective function is 5. Therefore the entry in c_B column corresponding to the new basic variable x_2 will be 5. Dividing all elements of the key row by the key element 3, the **new key row** is

$$5 \quad x_2 \quad \frac{8}{3} \quad \frac{2}{3} \quad 1 \quad 0 \quad \frac{1}{3} \quad 0 \quad 0$$

Transformation of R_2. Key column entry in R_2 is 2.

$\therefore$ $R_2(\text{new}) = R_2(\text{old}) - 2R_1(\text{new})$

$$10 - 2\left(\frac{8}{3}\right) = \frac{14}{3}$$
$$0 - 2\left(\frac{2}{3}\right) = -\frac{4}{3}$$
$$2 - 2(1) = 0$$
$$5 - 2(0) = 5$$
$$0 - 2\left(\frac{1}{3}\right) = -\frac{2}{3}$$
$$1 - 2(0) = 1$$
$$0 - 2(0) = 0$$

Transformation of R_3. Key column entry in R_3 is 2.

$\therefore$ $R_3(\text{new}) = R_3(\text{old}) - 2R_1(\text{new})$

$$15 - 2\left(\frac{8}{3}\right) = \frac{29}{3}$$
$$3 - 2\left(\frac{2}{3}\right) = \frac{5}{3}$$

$$2 - 2(1) = 0$$
$$4 - 2(0) = 4$$
$$0 - 2\left(\frac{1}{3}\right) = -\frac{2}{3}$$
$$0 - 2(0) = 0$$
$$1 - 2(0) = 1$$

The above information is summarized in the following table.

Simplex Table II

c_B	Basis	$c_j \rightarrow$ Solution $b(=x_B)$	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Ratio x_B/x_3
5	x_2	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	—
0	s_2	$\frac{14}{3}$	$-\frac{4}{3}$	0	5	$-\frac{2}{3}$	1	0	$\frac{\left(\frac{14}{3}\right)}{5} = \frac{14}{15} \rightarrow$
0	s_3	$\frac{29}{3}$	$\frac{5}{3}$	0	4	$-\frac{2}{3}$	0	1	$\frac{\left(\frac{29}{3}\right)}{4} = \frac{29}{12}$
$Z = \frac{40}{3}$		Z_j	$\frac{10}{3}$	5	0	$\frac{5}{3}$	0	0	
		$C_j = c_j - Z_j$	$-\frac{1}{3}$	0	4 $\uparrow$	$-\frac{5}{3}$	0	0	

$$* \quad Z = 5\left(\frac{8}{3}\right) + 0\left(\frac{14}{3}\right) + 0\left(\frac{29}{3}\right) = \frac{40}{3}$$

Since all entries in C_j-row are not negative or zero, the current solution is not optimal. The incoming variable is x_3 and the outgoing basic variable is s_2. Also 5 is the key number.

Step 5. The co-efficient of x_3 in the objective function is 4. Therefore the entry in c_B column corresponding to the new basic variable x_3 will be 4. Dividing all elements of the key row by the key element 5, the **new key row** is

$$4 \quad x_3 \quad \frac{14}{15} \quad -\frac{4}{15} \quad 0 \quad 1 \quad -\frac{2}{15} \quad \frac{1}{5} \quad 0$$

Transformation of R_1. Key column entry in R_1 is 0.

$\therefore$ $R_1(\text{new}) = R_1(\text{old}) - 0 \times R_2(\text{new})$
$= R_1(\text{odd})$

Transformation of R_3. Key column entry in R_3 is 4.

$\therefore$ $R_3(\text{new}) = R_3(\text{old}) - 4R_2(\text{new})$

$$\frac{29}{3} - 4\left(\frac{14}{15}\right) = \frac{89}{15}$$
$$\frac{5}{3} - 4\left(-\frac{4}{15}\right) = \frac{41}{15}$$

$$0 - 4(0) = 0$$
$$4 - 4(1) = 0$$
$$-\frac{2}{3} - 4\left(-\frac{2}{15}\right) = -\frac{2}{15}$$
$$0 - 4\left(\frac{1}{5}\right) = -\frac{4}{5}$$
$$1 - 4(0) = 1$$

The above information is summarized in the following table.

Simplex Table III

c_B	Basis	$c_j \rightarrow$ Solution $b(= x_B)$	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Ratio x_B/x_1
5	x_2	$\frac{8}{3}$	$\frac{2}{3}$	1	0	$\frac{1}{3}$	0	0	$\left(\frac{8}{3}\right)/\left(\frac{2}{3}\right) = 4$
4	x_3	$\frac{14}{15}$	$-\frac{4}{15}$	0	1	$-\frac{2}{15}$	$\frac{1}{5}$	0	—
0	s_3	$\frac{89}{15}$	$\frac{41}{15}$	0	0	$-\frac{2}{15}$	$-\frac{4}{5}$	1	$\left(\frac{89}{15}\right)/\left(\frac{41}{15}\right) = \frac{89}{41} \rightarrow$
$Z = \frac{256}{15}$		Z_j	$\frac{34}{15}$	5	4	$\frac{17}{15}$	$\frac{4}{5}$	0	
		$C_j = c_j - Z_j$	$\frac{11}{15}$ ↑	0	0	$-\frac{17}{15}$	$-\frac{4}{5}$	0	

$$* \quad Z = 5\left(\frac{8}{3}\right) + 4\left(\frac{14}{15}\right) + 0\left(\frac{89}{15}\right) = \frac{256}{15}$$

Since all entries in C_j-row are not negative or zero, the current sclution is not optimal.

The incoming variable is x_1 and the outgoing basic variable is s_3. Also $\frac{41}{15}$ is the key number.

Step 6. The co-efficient of x_1 in the objective function is 3. Therefore the entry in c_B column corresponding to the new basic variable x_1 will be 3. Dividing all elements of the key row by the key element $\frac{41}{15}$, the **new key row** is

3	x_1	$\frac{89}{41}$	1	0	0	$-\frac{2}{41}$	$-\frac{12}{41}$	$\frac{15}{41}$

Transformation of R_1. Key column entry in R_1 is $\frac{2}{3}$

$$\therefore \quad R_1(\text{new}) = R_1(\text{old}) - \frac{2}{3} R_3(\text{new})$$

$$\frac{8}{3} - \frac{2}{3}\left(\frac{89}{41}\right) = \frac{50}{41}$$

$$\frac{2}{3} - \frac{2}{3}(1) = 0$$
$$1 - \frac{2}{3}(0) = 1$$
$$0 - \frac{2}{3}(0) = 0$$
$$\frac{1}{3} - \frac{2}{3}\left(-\frac{2}{41}\right) = \frac{15}{41}$$
$$0 - \frac{2}{3}\left(-\frac{12}{41}\right) = \frac{8}{41}$$
$$0 - \frac{2}{3}\left(\frac{15}{41}\right) = -\frac{10}{41}$$

Transformation of R_2. Key column entry in R_2 is $-\frac{4}{15}$

$$\therefore \quad R_2(\text{new}) = R_2(\text{old}) + \frac{4}{15} R_3(\text{new})$$

$$\frac{14}{15} + \frac{4}{15}\left(\frac{89}{41}\right) = \frac{62}{41}$$
$$-\frac{4}{15} + \frac{4}{15}(1) = 0$$
$$0 + \frac{4}{15}(0) = 0$$
$$1 + \frac{4}{15}(0) = 1$$
$$-\frac{2}{15} + \frac{4}{15}\left(-\frac{2}{41}\right) = -\frac{6}{41}$$
$$\frac{1}{5} + \frac{4}{15}\left(-\frac{12}{41}\right) = \frac{5}{41}$$
$$0 + \frac{4}{15}\left(\frac{15}{41}\right) = \frac{4}{41}.$$

The above information is summarized in the following table.

Simplex Table IV

c_B	Basis	$c_j \rightarrow$ Solution $b(= x_B)$	3 x_1	5 x_2	4 x_3	0 s_1	0 s_2	0 s_3	Ratio
5	x_2	$\frac{50}{41}$	0	1	0	$\frac{15}{41}$	$\frac{8}{41}$	$-\frac{10}{41}$	
4	x_3	$\frac{62}{41}$	0	0	1	$-\frac{6}{41}$	$\frac{5}{41}$	$\frac{4}{41}$	
3	x_1	$\frac{89}{41}$	1	0	0	$-\frac{2}{41}$	$-\frac{12}{41}$	$\frac{15}{41}$	
$Z = \frac{765}{41}$		Z_j	3	5	4	$\frac{45}{41}$	$\frac{24}{41}$	$\frac{11}{41}$	
		$C_j = c_j - Z_j$	0	0	0	$-\frac{45}{41}$	$-\frac{24}{41}$	$-\frac{11}{41}$	

$$\therefore \quad Z = 5\left(\frac{50}{41}\right) + 4\left(\frac{62}{41}\right) + 3\left(\frac{89}{41}\right) = \frac{765}{41}$$

Since all entries in C_j-row are either negative or zero, therefore, the optimal solution is reached with

$$x_1 = \frac{89}{41},\ x_2 = \frac{50}{41},\ x_3 = \frac{62}{41} \text{ and Max. } Z = \frac{765}{41}$$

Example 3. *Solve the following linear programming problem by simplex method:*

Maximize $Z = 2x_1 + x_2 - x_3$

subject to

$$x_1 + x_2 \le 1$$
$$x_1 - 2x_2 - x_3 \ge -2$$
$$x_1, x_2, x_3 \ge 0.$$

Sol. Step 1. The problem is of maximization. As the sign of b in the second inequality is negative, it is first converted to positive by multiplying both the sides of the second inequality by (– 1). Thus, the problem can be restated as

Maximize $Z = 2x_1 + x_2 - x_3$

subject to

$$x_1 + x_2 \le 1$$
$$-x_1 + 2x_2 + x_3 \le 2$$
$$x_1, x_2, x_3 \ge 0.$$

Step 2. Introducing slack variables s_1, s_2 (one for each constraint) the problem in standard form is:

Maximize $Z = 2x_1 + x_2 - x_3 + 0s_1 + 0s_2$

subject to

$$x_1 + x_2 + s_1 = 1 \quad \ldots(1)$$
$$-x_1 + 2x_2 + x_3 + s_2 = 2 \quad \ldots(2)$$
$$x_1, x_2, x_3, s_1, s_2 \ge 0$$

Step 3. Since we have 2 equations in 5 variables, a solution is obtained by setting 5 – 2 = 3 variables equal to zero and solving for the remaining 2 variables. We choose initial basic feasible solution as:

$$x_1 = x_2 = x_3 = 0;\ s_1 = 1,\ s_2 = 2$$

at which Z = 0

Simplex Table I

		c_j	2	1	–1	0	0	Ratio
c_B	Basis	Solution $b(=x_B)$	x_1	x_2	x_3	s_1	s_2	x_B/x_1
0	s_1	1	1	1	0	1	0	$\frac{1}{1} = 1 \rightarrow$
0	s_2	2	–1	2	1	0	1	—
Z = 0		Z_j	0	0	0	0	0	
		$C_j = c_j - Z_j$	2 ↑	1	–1	0	0	

Step 4. Since some entries in C_j-row are positive, the current solution is not optimal. The incoming variable is x_1 and the outgoing basic variable is s_1. Also 1 is the key number.

Since the key number is 1, new key row is the same as old one.

Transformation of R_2. Key column entry in R_2 is (– 1)

$\therefore$

$R_2(\text{new}) = R_2(\text{old}) + 1\ R_1(\text{new})$
$2 + 1(1) = 3$
$-1 + 1(1) = 0$
$2 + 1(1) = 3$
$1 + 1(0) = 1$
$0 + 1(1) = 1$
$1 + 1(0) = 1$

The above information is summarized in the following table.

Simplex Table II

		$c_j \rightarrow$	2	1	–1	0	0	Ratio
c_B	Basis	Solution $b(=x_B)$	x_1	x_2	x_3	s_1	s_2	
2	x_1	1	1	1	0	1	0	
0	s_2	3	0	3	1	1	1	
Z = 2		Z_j	2	2	0	2	0	
		$C_j = c_j - Z_j$	0	–1	–1	–2	0	

Since all entries in C_j-row are either negative or zero, therefore, the optimal solution is reached with

$$x_1 = 1,\quad x_2 = 0,\quad x_3 = 0 \quad \text{and} \quad \text{Max. } Z = 2$$

Example 4. *Solve the following LPP by simplex method:*

Minimize $Z = x_1 - 3x_2 + 3x_3$

subject to

$$3x_1 - x_2 + 2x_3 \le 7$$
$$2x_1 + 4x_2 \ge -12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

(M.D.U. May 2011)

Sol. Step 1. The problem is that of minimization. Converting it into maximization problem by using the relationship

$$\text{Min. } Z = -\text{Max. } Z^* \qquad \text{where } Z^* = -Z \quad \ldots(1)$$

As the sign of b in the second inequality is negative, it is first converted to positive by multiplying both the sides of the second inequality by (– 1). Thus, the problem can be restated as

Maximize $Z^* = -x_1 + 3x_2 - 3x_3$

subject to

$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 - 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0.$$

Step 2. Introducing slack variables s_1, s_2, s_3 (one for each constraint), the problem in standard form is:

$$\text{Maximize } Z^* = -x_1 + 3x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$$

subject to

$$3x_1 - x_2 + 2x_3 + s_1 = 7$$
$$-2x_1 - 4x_2 + s_2 = 12$$
$$-4x_1 + 3x_2 + 8x_3 + s_3 = 10$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0.$$

Step 3. Since we have 3-equations in 6 variables, a solution is obtained by setting $6 - 3 = 3$ variables equal to zero and solving for the remaining 3 variables. We choose initial basic feasible solution as:

$$x_1 = x_2 = x_3 = 0;\ s_1 = 7,\ s_2 = 12,\ s_3 = 10 \text{ at which } Z^* = 0$$

Simplex Table I

c_B	Basis	$c_j \rightarrow$ Solution	-1	3	-3	0	0	0	Ratio
		$b(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	x_B/x_2
0	s_1	7	3	-1	2	1	0	0	—
0	s_2	12	-2	-4	0	0	1	0	—
0	s_3	10	-4	3	8	0	0	1	$\frac{10}{3} \rightarrow$
$Z^* = 0$		Z_j^*	0	0	0	0	0	0	
		$C_j = c_j - Z_j^*$	-1	3 $\uparrow$	-3	0	0	0	

Step 4. Since some entries in C_j – row are positive, the current solution is not optimal. The incoming variable is x_2 and the outgoing basic variable is s_3. Also the key number is 3.

Dividing all elements of the key row by the key number 3, the **new key row** is

$$3 \quad x_2 \quad \frac{10}{3} \quad -\frac{4}{3} \quad 1 \quad \frac{8}{3} \quad 0 \quad 0 \quad \frac{1}{3}$$

Transformation of R_1. Key column entry in R_1 is (-1)

$\therefore$ $R_1\text{ (new)} = R_1\text{(old)} + 1\ R_3\text{ (new)}$

$$7 + 1\left(\frac{10}{3}\right) = \frac{31}{3}$$
$$3 + 1\left(-\frac{4}{3}\right) = \frac{5}{3}$$
$$-1 + 1(1) = 0$$
$$2 + 1\left(\frac{8}{3}\right) = \frac{14}{3}$$

$$1 + 1(0) = 1$$
$$0 + 1(0) = 0$$
$$0 + 1\left(\frac{1}{3}\right) = \frac{1}{3}$$

Transformation of R_2. Key column entry in R_2 is (-4)

$\therefore$ $R_2\text{(new)} = R_2\text{(old)} + 4R_3\text{(new)}$

$$12 + 4\left(\frac{10}{3}\right) = \frac{76}{3}$$
$$-2 + 4\left(-\frac{4}{3}\right) = -\frac{22}{3}$$
$$-4 + 4\,(1) = 0$$
$$0 + 4\left(\frac{8}{3}\right) = \frac{32}{3}$$
$$0 + 4(0) = 0$$
$$1 + 4(0) = 1$$
$$0 + 4\left(\frac{1}{3}\right) = \frac{4}{3}$$

The above information is summarized in the following table.

Simplex Table II

c_B	Basis	$c_j \rightarrow$ Solution	-1	3	-3	0	0	0	Ratio
		$b(= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3	x_B/x_1
0	s_1	$\frac{31}{3}$	$\frac{5}{3}$	0	$\frac{14}{3}$	1	0	$\frac{1}{3}$	$\frac{31}{5} \rightarrow$
0	s_2	$\frac{76}{3}$	$-\frac{22}{3}$	0	$\frac{32}{3}$	0	1	$\frac{4}{3}$	—
3	x_2	$\frac{10}{3}$	$-\frac{4}{3}$	1	$\frac{8}{3}$	0	0	$\frac{1}{3}$	—
$Z^* = 10$		Z^*_j	-4	3	8	0	0	1	
		$C_j = c_j - Z^*_j$	3 $\uparrow$	0	-11	0	0	-1	

Step 5. Since all entries in C_j-row are not negative or zero, the current solution is not optimal.

The incoming variable is x_1 and the outgoing basic variable is s_1. Also the key number is $\frac{5}{3}$.

Dividing all elements of the key row by the key number $\frac{5}{3}$, the **new key row** is

$$-1 \quad x_1 \quad \frac{31}{5} \quad 1 \quad 0 \quad \frac{14}{5} \quad \frac{3}{5} \quad 0 \quad \frac{1}{5}$$

Transformation of R_2. Key column entry in R_2 is $-\frac{22}{3}$.

$$\therefore \quad R_2(\text{new}) = R_2(\text{old}) + \frac{22}{3} R_1(\text{new})$$

$$\frac{76}{3} + \frac{22}{3}\left(\frac{31}{5}\right) = \frac{354}{5}$$

$$-\frac{22}{3} + \frac{22}{3}(1) = 0$$

$$0 + \frac{22}{3}(0) = 0$$

$$\frac{32}{3} + \frac{22}{3}\left(\frac{14}{5}\right) = \frac{156}{5}$$

$$0 + \frac{22}{3}\left(\frac{3}{5}\right) = \frac{22}{5}$$

$$1 + \frac{22}{3}(0) = 1$$

$$\frac{4}{3} + \frac{22}{3}\left(\frac{1}{5}\right) = \frac{14}{5}$$

Transformation of R_3. Key column entry in R_3 is $-\frac{4}{3}$

$$\therefore \quad R_3(\text{new}) = R_3(\text{old}) + \frac{4}{3} R_1(\text{new})$$

$$\frac{10}{3} + \frac{4}{3}\left(\frac{31}{5}\right) = \frac{58}{5}$$

$$-\frac{4}{3} + \frac{4}{3}(1) = 0$$

$$1 + \frac{4}{3}(0) = 1$$

$$\frac{8}{3} + \frac{4}{3}\left(\frac{14}{5}\right) = \frac{32}{5}$$

$$0 + \frac{4}{3}\left(\frac{3}{5}\right) = \frac{4}{5}$$

$$0 + \frac{4}{3}(0) = 0$$

$$\frac{1}{3} + \frac{4}{3}\left(\frac{1}{5}\right) = \frac{3}{5}$$

The above information is summarized in the following table.

Simplex Table III

c_B	Basis	c_j Solution $b(=x_B)$	-1 x_1	3 x_2	-3 x_3	0 s_1	0 s_2	0 s_3	Ratio
-1	x_1	$\frac{31}{5}$	1	0	$\frac{14}{5}$	$\frac{3}{5}$	0	$\frac{1}{5}$	
0	s_2	$\frac{354}{5}$	0	0	$\frac{156}{5}$	$\frac{22}{5}$	1	$\frac{14}{5}$	
3	x_2	$\frac{58}{5}$	0	1	$\frac{32}{5}$	$\frac{4}{5}$	0	$\frac{3}{5}$	
$Z^* = \frac{143}{5}$		Z^*_j	-1	3	$\frac{82}{5}$	$\frac{9}{5}$	0	$\frac{8}{5}$	
		$C_j = c_j - Z^*_j$	0	0	$-\frac{97}{5}$	$-\frac{9}{5}$	0	$-\frac{8}{5}$	

Since all entries in C_j-row are either negative or zero, therefore, the optimal solution is reached with

$$x_1 = \frac{31}{5},\ x_2 = \frac{58}{5},\ x_3 = 0 \text{ and Max. } Z^* = \frac{143}{5}$$

Hence, from (1), Min. $Z = -\frac{143}{5}$.

Example 5. *Following data are available for a firm which manufactures three items A, B and C:*

Product	Time required in hours		Profit (in ₹)
	Assembly	Finishing	
A	10	2	80
B	4	5	60
C	5	4	30
Firm's Capacity	2000	1009	

Express the above data in the form of linear programming problem to maximize the profit from the production and solve it by simplex method.

Sol. Step 1. Let x_1, x_2, x_3 denote respectively the number of units of products A, B, C manufactured by the firm.

The mathematical formulation of the problem is:

Maximize $Z = 80x_1 + 60x_2 + 30x_3$

subject to the constraints

$$10x_1 + 4x_2 + 5x_3 \le 2000$$
$$2x_1 + 5x_2 + 4x_3 \le 1009$$
$$x_1, x_2, x_3 \ge 0$$

Step 2. Introducing slack variables s_1, s_2 (one for each constraint) the problem in standard form is:

Maximize $Z = 80x_1 + 60x_2 + 30x_3 + 0s_1 + 0s_2$

subject to
$$10x_1 + 4x_2 + 5x_3 + s_1 = 2000$$
$$2x_1 + 5x_2 + 4x_3 + s_2 = 1009$$
$$x_1, x_2, x_3, s_1, s_2 \ge 0$$

Step 3. Since we have 2 equations in 5 variables, a solution is obtained by setting 5 − 2 = 3 variables equal to zero and solving for the remaining 2 variables.

An initial basic feasible solution is obtained by setting $x_1 = x_2 = x_3 = 0$ so that $s_1 = 2000$, $s_2 = 1009$.

The initial simplex table corresponding to this solution is given below:

Simplex Table I

c_B	*Basis*	$c_j \rightarrow$ *Solution* $b(= x_B)$	80	60	30	0	0	*Ratio*
			x_1	x_2	x_3	s_1	s_2	x_B/x_1
0	s_1	2000	10	4	5	1	0	200 →
0	s_2	1009	2	5	4	0	1	$\frac{1009}{2}$
Z = 0		Z_j	0	0	0	0	0	
		$C_j = c_j - Z_j$	80 ↑	60	30	0	0	

Step 4. Since C_j-row has some positive entries, the current solution is not optimal.

The incoming variable is x_1 and the outgoing basic variable is s_1. The key number is 10.

Dividing each element in the key row *i.e.*, s_1-row, by the key number 10 the **new key row** is

$$80 \quad x_1 \quad 200 \quad 1 \quad \frac{2}{5} \quad \frac{1}{2} \quad \frac{1}{10} \quad 0$$

Transformation of R_2. Key column entry in R_2 is 2.

$\therefore$ $R_2(\text{new}) = R_2(\text{old}) - 2R_1(\text{new})$

$$1009 - 2(200) = 609$$
$$2 - 2(1) = 0$$

$$5 - 2\left(\frac{2}{5}\right) = \frac{21}{5}$$
$$4 - 2\left(\frac{1}{2}\right) = 3$$
$$0 - 2\left(\frac{1}{10}\right) = -\frac{1}{5}$$
$$1 - 2(0) = 1$$

The above information is summarized in the following table.

Simplex Table II

c_B	*Basis*	$c_j \rightarrow$ *Solution* $b(= x_B)$	80	60	30	0	0	*Ratio*
			x_1	x_2	x_3	s_1	s_2	x_B/x_2
80	x_1	200	1	$\frac{2}{5}$	$\frac{1}{2}$	$\frac{1}{10}$	0	500
0	s_2	609	0	$\frac{21}{5}$	3	$-\frac{1}{5}$	1	145 →
Z = 16000		Z_j	80	32	40	8	0	
		$C_j = c_j - Z_j$	0	28 ↑	− 10	− 8	0	

Step 5. Since C_j-row has a positive entry, the current solution is not optimal.

The incoming variable is x_2 and the outgoing basic variable is s_2. The key number is $\frac{21}{5}$.

Dividing each element in the key row by the key number $\frac{21}{5}$, the **new key row** is

$$60 \quad x_2 \quad 145 \quad 0 \quad 1 \quad \frac{5}{7} \quad -\frac{1}{21} \quad \frac{5}{21}$$

Transformation of R_1. Key column entry in R_1 is $\frac{2}{5}$.

$\therefore$ $R_1(\text{new}) = R_1(\text{old}) - \frac{2}{5} R_2(\text{new})$

$$200 - \frac{2}{5}(145) = 142$$
$$1 - \frac{2}{5}(0) = 1$$
$$\frac{2}{5} - \frac{2}{5}(1) = 0$$
$$\frac{1}{2} - \frac{2}{5}\left(\frac{5}{7}\right) = \frac{3}{14}$$

$$\frac{1}{10}-\frac{2}{5}\left(-\frac{1}{21}\right)=\frac{5}{42}$$

$$0-\frac{2}{5}\left(\frac{5}{21}\right)=-\frac{2}{21}$$

The above information is summarized in the following table.

Simplex Table III

c_B	Basis	$c_j \to$ Solution $b(=x_B)$	80 x_1	60 x_2	30 x_3	0 s_1	0 s_2	Ratio
80	x_1	142	1	0	$\frac{3}{14}$	$\frac{5}{42}$	$-\frac{2}{21}$	
60	x_2	145	0	1	$\frac{5}{7}$	$-\frac{1}{21}$	$\frac{5}{21}$	
Z = 20060		Z_j	80	60	60	$\frac{20}{3}$	$\frac{20}{3}$	
		$C_j = c_j - Z_j$	0	0	– 30	$-\frac{20}{3}$	$-\frac{20}{3}$	

Since all entries in C_j-row are either negative or zero, the optimal solution has been obtained with $x_1 = 142$, $x_2 = 145$ and Max. Z = 20060.

7.11. TIE FOR THE INCOMING VARIABLE (Key Column)

At any iteration, if two or more columns have the same largest positive value in C_j-row, a tie for the incoming variable occurs. In order to break this tie, the selection for key column (incoming variable) can be made arbitrarily. However, in order to minimize the number of iterations required to reach the optimal solution, the following rules may be used:

(*i*) If there is a tie between two decision variables, the choice can be made arbitrarily.

(*ii*) If there is a tie between two slack (or surplus) variables, the choice can be made arbitrarily.

(*iii*) If there is a tie between a decision variable and a slack (or surplus) variable, then the decision variable is selected as the incoming variable.

7.12. TIE FOR THE OUTGOING VARIABLE (Key Row)—DEGENERACY

In the simplex method, in order to determine which basic variable to go out of the basis (*i.e.*, to determine the key row), we find the ratio of entries in x_B-column by the corresponding (but positive) entries in the key column and select that variable to go out of the basis for which this ratio is the minimum. Sometimes this minimum ratio is not unique or values of one or more basic variables in the x_B-column are zero. This problem is known as **degeneracy** in linear programming and the basic feasible solution is **degenerate.** It is possible, in a degenerate situation, to obtain a sequence of simplex tables that correspond to basic feasible solutions which give the same value of the objective function. Moreover, we may eventually return to the first simplex table in the sequence. This is called **cycling.** When cycling occurs, it is possible

that we may never obtain the optimal value of the objective function. However, this phenomenon seldom occurs in practical problems.

Degeneracy may occur at any iteration of the simplex method. When there is a tie in the minimum ratios, the selection of the key row can be made arbitrarily. However, the number of iterations required to arrive at the optimal solution can be minimized by adopting the following rules.

(*i*) Identify the tied rows.

(*ii*) Divide the co-efficients of slack variables in each tied row by the corresponding positive number of the key column in the row, starting from left to right.

(*iii*) Compare the resulting ratios columnwise from left to right.

(*iv*) Select the row which has the smallest ratio. This row becomes the key row.

If the above ratios fail to break the tie, then find similar ratios for the decision variables.

Thus, the degeneracy is resolved. Simplex method is then continued until an optimal solution is obtained.

Example. *Using simplex method*

Maximize $Z = 5x_1 + 3x_2$

subject to

$$x_1 + x_2 \le 2$$
$$5x_1 + 2x_2 \le 10$$
$$3x_1 + 8x_2 \le 12$$
$$x_1, x_2 \ge 0.$$

(M.D.U. Dec., 2008; K.U.K. Dec. 2009, Dec. 2010)

Sol. Step 1. Introducing slack variables s_1, s_2, s_3 (one for each constraint) the problem in standard form is:

Maximize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + 0s_3$

subject to

$$x_1 + x_2 + s_1 = 2$$
$$5x_1 + 2x_2 + s_2 = 10$$
$$3x_1 + 8x_2 + s_3 = 12$$
$$x_1, x_2, s_1, s_2, s_3 \ge 0.$$

Step 2. Since we have 3 equations in 5 variables, a solution is obtained by setting 5 – 3 = 2 variables equal to zero and solving for the remaining 3 variables.

Thus an initial basic feasible solution is obtained by setting $x_1 = x_2 = 0$ so that $s_1 = 2$, $s_2 = 10$, $s_3 = 12$.

The initial simplex table corresponding to this solution is given below:

Simplex Table I

c_B	Basis	$c_j \to$ Solution $b(=x_B)$	5 x_1	3 x_2	0 s_1	0 s_2	0 s_3	Ratio x_B/x_1
0	s_1	2	1	1	1	0	0	2 } Tie
0	s_2	10	5	2	0	1	0	2 } Tie
0	s_3	12	3	8	0	0	1	4
Z = 0		Z_j	0	0	0	0	0	
		$C_j = c_j - Z_j$	5 ↑	3	0	0	0	

Since C_j-row has some positive entries, the current solution is not optimal. The incoming variable is x_1.

Step 3. From table I, we observe that there is a tie among the s_1-row and s_2-row. This is an indication of the existence of degeneracy.

To obtain the unique key row, we adop the following procedure:

(*i*) The co-efficients of slack variables s_1 and s_2 are:

Row	*Column*	
	s_1	s_2
s_1	1	0
s_2	0	1

(*ii*) Key column element in s_1-row is 1 and in s_2-row is 5.

Dividing the co-efficients in s_1-row by 1 and in s_2-row by 5, we get the following ratios:

Row	*Column*	
	s_1	s_2
s_1	$\frac{1}{1}=1$	$\frac{0}{1}=0$
s_2	$\frac{0}{5}=0$	$\frac{1}{5}$

(*iii*) Comparing the ratios of Step (*ii*) from left to right columnwise, we find that in the first column, the second row yields the smaller ratio and, therefore, s_2-row becomes the key row and s_2 is the outgoing basic variable. The key number is 5.

Step 4. Dividing each element in the key row by the key number 5, the **new key row** is

$$5 \quad x_1 \quad 2 \quad 1 \quad \frac{2}{5} \quad 0 \quad \frac{1}{5} \quad 0$$

Transformation of R_1. Key column entry in R_1 is 1

$$\therefore \quad R_1(\text{new}) = R_1(\text{old}) - 1\, R_2(\text{new})$$

$$\begin{aligned} 2-1(2) &= 0 \\ 1-1(1) &= 0 \\ 1-1\left(\frac{2}{5}\right) &= \frac{3}{5} \\ 1-1(0) &= 1 \\ 0-1\left(\frac{1}{5}\right) &= -\frac{1}{5} \\ 0-1(0) &= 0 \end{aligned}$$

Transformation of R_3. Key column entry in R_3 is 3

$$\therefore \quad R_3(\text{new}) = R_3(\text{old}) - 3\, R_2(\text{new})$$

$$\begin{aligned} 12-3(2) &= 6 \\ 3-3(1) &= 0 \end{aligned}$$

$$\begin{aligned} 8-3\left(\frac{2}{5}\right) &= \frac{34}{5} \\ 0-3(0) &= 0 \\ 0-3\left(\frac{1}{5}\right) &= -\frac{3}{5} \\ 1-3(0) &= 1 \end{aligned}$$

The above information is summarized in the following table.

Simplex Table II

c_B	*Basis*	$c_j \rightarrow$ *Solution* $b(=x_B)$	5 x_1	3 x_2	0 s_1	0 s_2	0 s_3	*Ratio* x_B/x_2
0	s_1	0	0	$\frac{3}{5}$	1	$-\frac{1}{5}$	0	$0 \rightarrow$
5	x_1	2	1	$\frac{2}{5}$	0	$\frac{1}{5}$	0	5
0	s_3	6	0	$\frac{34}{5}$	0	$-\frac{3}{5}$	1	$\frac{15}{17}$
Z = 10		Z_j	5	2	0	1	0	
		$C_j = c_j - Z_j$	0	1 ↑	0	−1	0	

Since C_j-row has a positive entry, the current solution is not optimal. The incoming variable is x_2 and the outgoing basic variable is s_1. The key number is $\frac{3}{5}$.

Dividing each element in the key row by the key number $\frac{3}{5}$, the **new key row** is

$$3 \quad x_2 \quad 0 \quad 0 \quad 1 \quad \frac{5}{3} \quad -\frac{1}{3} \quad 0$$

Transformation of R_2. Key column entry in R_2 is $\frac{2}{5}$

$$\therefore \quad R_2(\text{new}) = R_2(\text{old}) - \frac{2}{5}\, R_1(\text{new})$$

$$\begin{aligned} 2-\frac{2}{5}(0) &= 2 \\ 1-\frac{2}{5}(0) &= 1 \\ \frac{2}{5}-\frac{2}{5}(1) &= 0 \\ 0-\frac{2}{5}\left(\frac{5}{3}\right) &= -\frac{2}{3} \end{aligned}$$

$$\frac{1}{5} - \frac{2}{5}\left(-\frac{1}{3}\right) = \frac{1}{3}$$

$$0 - \frac{2}{5}\,(0) \quad = 0$$

Transformation of R_3. Key column entry in R_3 is $\frac{34}{5}$

$$\therefore \qquad R_3(\text{new}) = R_3(\text{old}) - \frac{34}{5}\, R_1(\text{new})$$

$$6 - \frac{34}{5}\,(0) \quad = 6$$

$$0 - \frac{34}{5}\,(0) \quad = 0$$

$$\frac{34}{5} - \frac{34}{5}\,(1) = 0$$

$$0 - \frac{34}{5}\left(\frac{5}{3}\right) = -\frac{34}{3}$$

$$-\frac{3}{5} - \frac{34}{5}\left(-\frac{1}{3}\right) = \frac{5}{3}$$

$$1 - \frac{34}{5}\,(0) \quad = 1$$

The above information is summarized in the following table.

Simplex Table III

		$c_j \rightarrow$	5	3	0	0	0	*Ratio*
c_B	*Basis*	*Solution* $b(= x_B)$	x_1	x_2	s_1	s_2	s_3	
3	x_2	0	0	1	$\frac{5}{3}$	$-\frac{1}{3}$	0	
5	x_1	2	1	0	$-\frac{2}{3}$	$\frac{1}{3}$	0	
0	s_3	6	0	0	$-\frac{34}{3}$	$\frac{5}{3}$	1	
Z = 10		Z_j	5	3	$\frac{5}{3}$	$\frac{2}{3}$	0	
		$C_j = c_j - Z_j$	0	0	$-\frac{5}{3}$	$-\frac{2}{3}$	0	

Since all entries in C_j-row are either negative or zero, the optimal solution has been obtained with $x_1 = 2$, $x_2 = 0$ and Max. Z = 10.

EXERCISE 7.5

Using simplex method, solve the following linear programming problems (1–6):

1. (*i*) Maximize $Z = x_1 + 2x_2$
   subject to
   $$2x_1 + x_2 \le 8$$
   $$2x_1 + 3x_2 \le 12$$
   $$x_1, x_2 \ge 0$$

   (*ii*) Maximize $Z = x_1 + 3x_2$ (M.D.U. Dec. 2011)
   subject to
   $$x_1 + 2x_2 \le 10$$
   $$0 \le x_1 \le 5$$
   $$0 \le x_2 \le 4$$

   (*iii*) Maximize $Z = 6x_1 + 4x_2$
   subject to
   $$x_1 + 2x_2 \le 240$$
   $$3x_1 + 2x_2 \le 300$$
   $$x_1, x_2 \ge 0$$

   (*iv*) Maximize $Z = 4x_1 + 5x_2$
   subject to
   $$x_1 - 2x_2 \le 2$$
   $$2x_1 + x_2 \le 6$$
   $$x_1 + 2x_2 \le 5$$
   $$-x_1 + x_2 \le 2$$
   $$x_1, x_2 \ge 0$$

   (*v*) Maximize $Z = 45x_1 + 80x_2$
   subject to
   $$5x_1 + 20x_2 \le 400$$
   $$10x_1 + 15x_2 \le 450$$
   $$x_1, x_2 \ge 0$$

2. (*i*) Maximize $Z = 10x_1 + x_2 + 2x_3$
   subject to
   $$x_1 + x_2 - 2x_3 \le 10$$
   $$4x_1 + x_2 + x_3 \le 20$$
   $$x_1, x_2, x_3 \ge 0$$
   (M.D.U. May 2009)

   (*ii*) Maximize $Z = 5x_1 + 4x_2 + 3x_3$
   subject to
   $$3x_1 + 2x_2 + x_3 \le 10$$
   $$2x_1 + x_2 + 2x_3 \le 12$$
   $$x_1 + x_2 + 3x_3 \le 15$$
   $$x_1, x_2, x_3 \ge 0$$

   (*iii*) Maximize $Z = x_1 + x_2 + 3x_3$
   subject to
   $$3x_1 + 2x_2 + x_3 \le 3$$
   $$2x_1 + x_2 + 2x_3 \le 2$$
   $$x_1, x_2, x_3 \ge 0$$

   (*iv*) Maximize $Z = 4x_1 + 3x_2 + 4x_3 + 6x_4$
   subject to
   $$x_1 + 2x_2 + 2x_3 + 4x_4 \le 80$$
   $$2x_1 + 2x_3 + x_4 \le 60$$
   $$3x_1 + 3x_2 + x_3 + x_4 \le 80$$
   $$x_1, x_2, x_3, x_4 \ge 0$$

3. Maximize $Z = x_1 + 2x_2 + x_3$
   subject to
   $$2x_1 + x_2 - x_3 \le 2$$
   $$-2x_1 + x_2 - 5x_3 \ge -6$$

$$4x_1 + x_2 + x_3 \le 6$$
$$x_1, x_2, x_3 \ge 0$$

4. Minimize $Z = x_1 - 3x_2 + 2x_3$
subject to

$$3x_1 - x_2 + 2x_3 \le 7$$
$$-2x_1 + 4x_2 \le 12$$
$$-4x_1 + 3x_2 + 8x_3 \le 10$$
$$x_1, x_2, x_3 \ge 0$$

(*M.D.U. May 2006*)

5. (*i*) Maximize $Z = 5x_1 + 6x_2 + x_3$
subject to

$$9x_1 + 3x_2 - 2x_3 \le 5$$
$$4x_1 + 2x_2 - x_3 \le 2$$
$$x_1 - 4x_2 + x_3 \le 3$$
$$x_1, x_2, x_3 \ge 0$$

(*ii*) Maximize $Z = x_1 + 4x_2 - x_3$
subject to

$$-5x_1 + 6x_2 - 2x_3 \le 30$$
$$-x_1 + 3x_2 + 6x_3 \le 12$$
$$x_1, x_2, x_3 \ge 0$$

6. (*i*) Maximize $Z = 3x_1 + 9x_2$
subject to

$$x_1 + 4x_2 \le 8$$
$$x_1 + 2x_2 \le 4$$
$$x_1, x_2 \ge 0$$

(*ii*) Maximize $Z = 50x_1 + 60x_2 + 80x_3$
subject to

$$2x_1 + x_2 + 2x_3 \le 50$$
$$x_1 + 6x_2 + 2x_3 \le 50$$
$$x_1 + 2x_2 + x_3 \le 26$$
$$x_1, x_2, x_3 \ge 0$$

(*iii*) Maximize $Z = 107x_1 + x_2 + 2x_3$
subject to the constraints

$$14x_1 + x_2 - 6x_3 + 3x_4 = 7$$
$$16x_1 + \frac{1}{2}x_2 - 6x_3 \le 5$$
$$3x_1 - x_2 - x_3 \le 0$$
$$x_1, x_2, x_3, x_4 \ge 0$$

(*M.D.U. Dec. 2010*)

7. A firm makes two types of furnitures, chairs and tables. Profits are ₹ 20 per chair and ₹ 30 per table. Both products are processed on three machines M_1, M_2 and M_3. The time required for each product in hours and total time available in hours per week on each machine are as follows:

Machine	*Chair*	*Table*	*Available Time*
M_1	3	3	36
M_2	5	2	50
M_3	2	6	60

How should the manufacturer schedule his production in order to maximize profit?
(Use simplex method)

8. A company makes two kinds of leather belts. Belt A is high quality belt and belt B is of lower quality. The respective profits are ₹ 4 and ₹ 3 per belt. The production of each of type A requires twice as much time as a belt of type B, and if all belts were of type B, the company could make 1000 per day. The supply of leather is sufficient for only 800 belts per day (both A and B combined). Belt A requires a fancy buckle and only 400 per day are available. There are only 700 buckles a day available for belt B.

What should be the daily production of each type of belt? Formulate this problem as an LP model and solve it by simplex method.

9. A farmer has 1000 acres of land on which he can grow corn, wheat or soyabeans. Each acre of corn costs ₹ 100 for preparation, requires 7 man days of work and yields a profit of ₹ 30. An acre of wheat costs ₹ 120 to prepare, requires 10 man days of work and yields a profit of ₹ 40. An acre of soyabeans costs ₹ 70 to prepare, requires 8 man days of work and yields a profit of ₹ 20. If the farmer has ₹ 1,00,000 for preparation and can count on 8000 man days of work, how many acres should be allocated to each crop to maximize profits. Formulate this as an LPP and solve it by simplex method.

10. A manufacturing firm has discontinued production of a certain unprofitable product line. This has created considerable excess production capacity. Management is considering to devote this excess capacity to one or more of three products 1, 2 and 3. The available capacity on machines and the number of machine-hours required for each unit of the respective product, is given below:

Machine Type	*Available Time (in Machine-hours per week)*	*Productivity (in Machine-hours per unit)*		
		Product 1	*Product 2*	*Product 3*
Milling Machine	250	8	2	3
Lathe	150	4	3	0
Grinder	50	2	—	1

The profit per unit would be ₹ 20, ₹ 6 and ₹ 8 respectively for product 1, 2 and 3. Find how much of each product the firm should produce in order to maximize profit?

11. A company can make three different products A, B and C by combining steel and rubber. Product A requires 2 units of steel and 3 units of rubber and can be sold at a profit of ₹ 40 per unit. Product B requires 3 units of steel and 3 units of rubber and can be sold at a profit of ₹ 45 per unit. Product C requires 1 unit of steel and 2 units of rubber and yields ₹ 24 profit per unit. There are 100 units of steel and 120 units of rubber available per day. What should be the daily production of each of the three products in order to maximize profit? Use simplex method.

Answers

1. (*i*) $x_1 = 0,\ x_2 = 4$; Max. $Z = 8$ (*ii*) $x_1 = 2,\ x_2 = 4$; Max. $Z = 14$
(*iii*) $x_1 = 100,\ x_2 = 0$; Max. $Z = 600$ (*iv*) $x_1 = \frac{7}{3},\ x_2 = \frac{4}{3}$; Max. $Z = 16$
(*v*) $x_1 = 24,\ x_2 = 14$; Max. $Z = 2200$
2. (*i*) $x_1 = 5,\ x_2 = x_3 = 0$; Max. $Z = 50$ (*ii*) $x_1 = 0,\ x_2 = 3,\ x_3 = 4$; Max. $Z = 24$
(*iii*) $x_1 = x_2 = 0,\ x_3 = 1$; Max. $Z = 3$
(*iv*) $x_1 = \frac{280}{13},\ x_2 = 0,\ x_3 = \frac{20}{13},\ x_4 = \frac{180}{13}$; Max. $Z = \frac{2280}{13}$
3. $x_1 = 0,\ x_2 = 4,\ x_3 = 2$; Max. $Z = 10$ 4. $x_1 = 4,\ x_2 = 5,\ x_3 = 0$; Min. $Z = -11$
5. (*i*) Unbounded solution (*ii*) Unbounded solution
[**Hint.** When each entry in the key column is negative or zero, Z is unbounded]
6. (*i*) $x_1 = 0,\ x_2 = 2$; Max. $Z = 18$ (*ii*) $x_1 = x_2 = 0,\ x_3 = 25$; Max. $Z = 2000$
(*iii*) Unbounded solution. [Here x_4 is also a slack variable]
7. 3 chairs and 9 tables per week, maximum profit = ₹ 330
[**Hint.** Let x and y be the number of chairs and tables to be produced per week. Then
Maximize $Z = 20x + 30y$
subject to

$$3x + 3y \le 36$$
$$5x + 2y \le 50$$

$$2x + 6y \le 60$$
$$x, y \ge 0$$

8. Let x_1 and x_2 be the number of belts of type A and B respectively manufactured each day. Then the mathematical LP model is:

Maximize $Z = 4x_1 + 3x_2$

subject to
$$2x_1 + x_2 \le 1000$$
$$x_1 + x_2 \le 800$$
$$x_1 \le 400$$
$$x_2 \le 700$$
$$x_1, x_2 \ge 0$$

Optimal solution is $x_1 = 200$, $x_2 = 600$; max. Z = ₹ 2,600.

9. Let x_1, x_2, x_3 acres of land be allocated to corn, wheat and soyabeans respectively. Then

Maximize $Z = 30x_1 + 40x_2 + 20x_3$

subject to
$$100x_1 + 120x_2 + 70x_3 \le 1,00,000$$
$$7x_1 + 10x_2 + 8x_3 \le 8000$$
$$x_1 + x_2 + x_3 \le 1000$$
$$x_1, x_2, x_3 \ge 0$$

Optimal solution is $x_1 = 250$, $x_2 = 625$, $x_3 = 0$; max. Z = ₹ 32500.

10. Let x_1, x_2, x_3 be the number of units of product 1, 2 and 3 to be produced per week. Then

Maximize $Z = 20x_1 + 6x_2 + 8x_3$

subject to
$$8x_1 + 2x_2 + 3x_3 \le 250$$
$$4x_1 + 3x_2 \le 150$$
$$2x_1 + x_3 \le 50$$
$$x_1, x_2, x_3 \ge 0$$

Optimal solution: $x_1 = 0$, $x_2 = 50$, $x_3 = 50$; Max. Z = 700.

11. 20 units of A, 20 units of B, 0 units of C with max. profit = ₹ 1700.

7.13. ARTIFICIAL VARIABLES

In the linear programming problems discussed so far, we have the following characteristics:

(*i*) The objective function is of maximization type

(*ii*) All constraints are of $\le$ form

(*iii*) Right-hand side of each constraint is positive.

Adding slack variables, all inequality constraints are converted into equalities. Initial solution is found very conveniently by letting the slack variables be the initial basic variables so that each one equals the positive right hand side of its equation.

All linear programming problems donot have the above characteristics. Now we will see what adjustments are required for other forms ($\ge$, $=$ or $b_i < 0$) of linear programming problems.

When the constraints are of $\le$ type but some $b_i < 0$, after adding the non-negative slack variable s_i, the initial solution will involve $s_i = b_i < 0$. This solution is not feasible because it violates the non-negativity condition of slack variable.

When the constraints are of $\ge$ type, after adding the non-negative surplus variable s_i and letting each decision variable equal to zero, we get an initial solution $-s_i = b_i$ or $s_i = -b_i < 0$. This solution is not feasible because it violates the non-negativity condition of surplus variable.

Thus, introduction of slack/surplus variables fails to give a basic solution in many linear programming problems. Such problems are solved by the **artificial variable technique** explained below:

7.14. SIMPLEX METHOD (Minimization Case)

Consider the general linear programming problem:

Minimize $Z = c_1x_1 + c_2x_2 + ... + c_nx_n$

subject to the constraints
$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n \ge b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n \ge b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n \ge b_m$$

and
$$x_1, x_2, \ldots\ldots, x_n \ge 0$$

Subtracting surplus variables s_i to convert inequalities into equalities, the above problem reduces to:

Minimize $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n + 0s_1 + 0s_2 + \ldots\ldots + 0s_m$

subject to the constraints
$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n - s_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n - s_2 = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n - s_m = b_m$$

and
$$x_j \ge 0, \quad (j = 1, 2, ..., n)$$
$$s_i \ge 0, \quad (i = 1, 2, ..., m)$$

An initial basic solution is obtained by assigning zero value to each decision variable *i.e.*, by setting
$$x_1 = x_2 = \ldots\ldots = x_n = 0$$

Thus, we get
$$-s_1 = b_1 \quad \text{or} \quad s_1 = -b_1$$
$$-s_2 = b_2 \quad \text{or} \quad s_2 = -b_2$$
$$\vdots \qquad\qquad \vdots$$
$$-s_m = b_m \quad \text{or} \quad s_m = -b_m$$

which is not feasible because it violates the non-negativity constraints $s_i \ge 0$.

$\therefore$ The simplex algorithm needs modification. We now introduce m new variables $A_1, A_2, ..., A_m$ into the system of constraints.

These new variables are called **artificial variables.** The resulting system of equations can now be written as
$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n - s_1 + A_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n - s_2 + A_2 = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n - s_m + A_m = b_m$$

where
$$x_j \ge 0, \quad (j = 1, 2, ..., n)$$

and
$$s_i \ge 0, \quad A_i \ge 0, (i = 1, 2, ..., m)$$

Thus the standard LPP has been reduced to a system of m equations in $(n + 2m)$ variables [n decision variables + m surplus variables + m artificial variables]. An initial basic feasible solution can now be obtained by setting $(n + 2m) - m = n + m$ variables equal to zero (*i.e.*, by setting each decision variable and each surplus variable equal to zero). Thus the initial basic feasible solution is given by $A_1 = b_1, A_2 = b_2, \ldots\ldots, A_m = b_m$.

This, however, does not constitute a solution to the original system of equations because the two systems are not equivalent. To remove artificial variables from solution, we use

Big-M Method or Method of Penalties

We assign a zero co-efficient to surplus variables and a very large positive co-efficient +M to artificial variable in the objective function.

Therefore, the problem can now be re-written as follows:

Minimize $Z = c_1x_1 + c_2x_2 + \ldots\ldots + c_nx_n + 0s_1 + 0s_2 + \ldots + 0s_m + MA_1 + MA_2 + \ldots + MA_m$

subject to the constraints

$$a_{11}x_1 + a_{12}x_2 + \ldots\ldots + a_{1n}x_n - s_1 + A_1 = b_1$$
$$a_{21}x_1 + a_{22}x_2 + \ldots\ldots + a_{2n}x_n - s_2 + A_2 = b_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots + a_{mn}x_n - s_m + A_m = b_m$$

and $\quad x_j \geq 0; (j = 1, 2, \ldots, n)$

$s_i \geq 0, \quad A_i \geq 0; (i = 1, 2, \ldots, m)$

Solve the modified LPP by simplex method.

Test for optimality. Compute the values of $C_j = c_j - Z_j$ in the last row of the simplex table.

(*i*) If all entries in C_j-row are non-negative (*i.e.*, ≥ 0), then the current basic feasible solution is optimal.

(*ii*) If C_k is most negative [*i.e.*, $C_k = \min. \{C_j : C_j < 0\}$] in a column and all entries in this column are negative, then the problem has an unbounded optimal solution.

(*iii*) If C_j has one or more negative values, then select the variable to enter into the basis with the largest negative value, *i.e.*, select x_k when $C_k = \min. \{C_j : C_j < 0\}$.

Determine the key row and key element in the same manner as discussed in the simplex algorithm of the maximization case.

Remarks 1. At any iteration of simplex method, one of the following three cases may arise:

(*i*) There remains no artificial variable in the basis and the optimality condition is satisfied, then the solution is an optimal basic feasible solution to the problem.

(*ii*) There is at least one artificial variable present in the basis with zero value and the co-efficient of M in each C_j value is non-negative, then the given LPP has no solution *i.e.*, the current basic feasible solution is degerate.

(*iii*) There is at least one artificial variable in the basis with positive value and the co-efficient of M in each C_j-value is non-negative, then the given LPP has no optimal basic feasible solution. In this case, the given LPP is said to have a **pseudo optimal basic feasible solution.**

2. The artificial variables are fictitious. They are introduced only for computational purposes and donot have any physical meaning or economic significance.

3. The constraints with $\geq$ inequality sign require a surplus variable and an artificial variable.

The constraints with '=' sign require neither a slack variable nor a surplus variable. Only an artificial variable has to be added to get the initial solution.

4. By assigning a very large per unit penalty to each of the artificial variables in the objective function, these variables can be removed from the simplex table as soon as they become non-basic. Such a penalty is designated as – M for maximization problems and + M for minimization problems where M > 0.

5. *No feasible solution* exists to the problem if all the artificial variables cannot be driven out in the optimum Big-M simplex table. Infeasibility is due to the presence of inconsistent constraints in the formulation of LP problems where the resources are not sufficient to meet the expected demands.

6. In minimization problems, key column is the one with largest negative number in C_j-row.

In maximization problems, key column is the one with largest positive number in C_j-row.

7. It must be noted that all variables, except the artificial variables, once driven out in an iteration, may re-enter in a subsequent iteration. But an artificial variable, once driven out, can never re-enter. Therefore, in the iteration subsequent to the one from which an artificial variable is drawn out, do not compute the column for the particular artificial variable.

ILLUSTRATIVE EXAMPLES

Example 1. *Use the penalty (Big-M) method to solve the following LP problem:*

Minimize $Z = 5x_1 + 3x_2$

subject to the constraints

$$2x_1 + 4x_2 \leq 12$$
$$2x_1 + 2x_2 = 10$$
$$5x_1 + 2x_2 \geq 10$$

and $\quad x_1, x_2 \geq 0$

Sol. Step 1. The first constraint involves $\leq$. We introduce only a slack variable s_1 thereby getting

$$2x_1 + 4x_2 + s_1 = 12$$

The second constraint is strict equality and requires neither a slack variable nor a surplus variable. We add only an artificial variable A_1 thereby getting

$$2x_1 + 2x_2 + A_1 = 10$$

The third constraint involves $\geq$. We introduce a surplus variable s_2 and an artificial variable A_2 thereby getting

$$5x_1 + 2x_2 - s_2 + A_2 = 10$$

$\therefore$ The standard form of the LP problem now becomes:

Minimize $Z = 5x_1 + 3x_2 + 0s_1 + 0s_2 + MA_1 + MA_2$

subject to the constraints

$$2x_1 + 4x_2 + s_1 = 12$$
$$2x_1 + 2x_2 + A_1 = 10$$
$$5x_1 + 2x_2 - s_2 + A_2 = 10$$

and $\quad x_1, x_2, s_1, s_2, A_1, A_2 \geq 0$

Step 2. Since we have 3 equations in 6 variables, a solution is obtained by setting $6 - 3 = 3$ variables equal to zero and solving for the remaining 3 variables. An initial basic feasible solution is obtained by setting $x_1 = x_2 = s_2 = 0$. Therefore, the initial basic feasible solution is $s_1 = 12$, $A_1 = 10$, $A_2 = 10$ and $Z = 10\text{ M} + 10\text{ M} = 20\text{ M}$. The initial basic feasible solution is given in the simplex table below.

Simplex Table I

c_B	Basis	$c_j \to$ Solution $b(=x_B)$	5 x_1	3 x_2	0 s_1	0 s_2	M A_1	M A_2	Ratio x_B/x_1
0	s_1	12	2	4	1	0	0	0	$\frac{12}{2}=6$
M	A_1	10	2	2	0	0	1	0	$\frac{10}{2}=5$
M	A_2	10	5	2	0	-1	0	1	$\frac{10}{5}=2 \to$
Z = 20M		Z_j	7 M	4 M	0	$-$M	M	M	
		$C_j = c_j - Z_j$	$5 - 7$ M ↑	$3 - 4$M	0	M	0	0	

Some entries in C_j-row being negative, the current solution is not optimal.

Step 3. Largest negative entry in C_j-row is $5 - 7$ M which lies in x_1-column. Therefore the incoming variable is x_1. The ratio 2 is minimum in A_2-row, therefore, the outgoing basic variable is A_2. (In further simplex tables, we will not compute the A_2-column). Key element is 5.

New key row is

$$5 \quad x_1 \quad 2 \quad 1 \quad \frac{2}{5} \quad 0 \quad -\frac{1}{5} \quad 0$$

Transformation of R_1. Key column entry in R_1 is 2.

$\therefore$ $R_1(\text{new}) = R_1(\text{old}) - 2\,R_3(\text{new})$

$$12 - 2(2) = 8$$
$$2 - 2(1) = 0$$
$$4 - 2\left(\frac{2}{5}\right) = \frac{16}{5}$$
$$1 - 2(0) = 1$$
$$0 - 2\left(-\frac{1}{5}\right) = \frac{2}{5}$$
$$0 - 2(0) = 0$$

Transformation of R_2. Key column entry in R_2 is 2.

$\therefore$ $R_2(\text{new}) = R_2(\text{old}) - 2\,R_3(\text{new})$

$$10 - 2(2) = 6$$
$$2 - 2(1) = 0$$
$$2 - 2\left(\frac{2}{5}\right) = \frac{6}{5}$$
$$0 - 2(0) = 0$$

$$0 - 2\left(-\frac{1}{5}\right) = \frac{2}{5}$$
$$1 - 2(0) = 1$$

New simplex table is given below.

Simplex Table II

c_B	Basis	$c_j \to$ Solution $b(=x_B)$	5 x_1	3 x_2	0 s_1	0 s_2	M A_1	Ratio x_B/x_2
0	s_1	8	0	$\frac{16}{5}$	1	$\frac{2}{5}$	0	$\frac{5}{2} \to$
M	A_1	6	0	$\frac{6}{5}$	0	$\frac{2}{5}$	1	5
5	x_1	2	1	$\frac{2}{5}$	0	$-\frac{1}{5}$	0	5
Z = 10 + 6 M		Z_j	5	$2 + \frac{6M}{5}$	0	$-1 + \frac{2M}{5}$	M	
		$C_j = c_j - Z_j$	0	$1 - \frac{6M}{5}$ ↑	0	$1 - \frac{2M}{5}$	0	

Some entries in C_j-row being negative, the current solution is not optimal.

Step 4. Largest negative entry in C_j-row is $1 - \frac{6M}{5}$ which lies in x_2-column. Therefore the incoming variable is x_2. The ratio $\frac{5}{2}$ is minimum in s_1-row, therefore, the outgoing basic variable is s_1. Key element is $\frac{16}{5}$.

New key row is

$$3 \quad x_2 \quad \frac{5}{2} \quad 0 \quad 1 \quad \frac{5}{16} \quad \frac{1}{8} \quad 0$$

Transformation of R_2. Key column entry in R_2 is $\frac{6}{5}$

$\therefore$ $R_2(\text{new}) = R_2(\text{old}) - \frac{6}{5}\,R_1(\text{new})$

$$6 - \frac{6}{5}\left(\frac{5}{2}\right) = 3$$
$$0 - \frac{6}{5}(0) = 0$$
$$\frac{6}{5} - \frac{6}{5}(1) = 0$$

$$0-\frac{6}{5}\left(\frac{5}{16}\right)=-\frac{3}{8}$$

$$\frac{2}{5}-\frac{6}{5}\left(\frac{1}{8}\right)=\frac{1}{4}$$

$$1-\frac{6}{5}(0)=1$$

Transformation of R_3. Key column entry in R_3 is $\frac{2}{5}$.

$$\therefore \quad R_3(\text{new}) = R_3(\text{old}) - \frac{2}{5}\,R_1(\text{new})$$

$$2-\frac{2}{5}\left(\frac{5}{2}\right)=1$$

$$1-\frac{2}{5}(0)=1$$

$$\frac{2}{5}-\frac{2}{5}(1)=0$$

$$0-\frac{2}{5}\left(\frac{5}{16}\right)=-\frac{1}{8}$$

$$-\frac{1}{5}-\frac{2}{5}\left(\frac{1}{8}\right)=-\frac{1}{4}$$

$$0-\frac{2}{5}(0)=0$$

The new simplex table is given below.

Simplex Table III

		$c_j \rightarrow$ Solution	5	3	0	0	M	Ratio
c_B	Basis	$b(=x_B)$	x_1	x_2	s_1	s_2	A_1	x_B/s_2
3	x_2	$\frac{5}{2}$	0	1	$\frac{5}{16}$	$\frac{1}{8}$	0	$\frac{(5/2)}{(1/8)}=20$
M	A_1	3	0	0	$-\frac{3}{8}$	$\frac{1}{4}$	1	$\frac{3}{\left(\frac{1}{4}\right)}=12 \rightarrow$
5	x_1	1	1	0	$-\frac{1}{8}$	$-\frac{1}{4}$	0	—
$Z=\frac{25}{2}+3M$		Z_j	5	3	$\frac{5}{16}-\frac{3M}{8}$	$-\frac{7}{8}+\frac{M}{4}$	M	
		$C_j=c_j-Z_j$	0	0	$-\frac{5}{16}+\frac{3M}{8}$	$\frac{7}{8}-\frac{M}{4}$ ↑	0	

Since one entry in C_j-row is negative, the current solution is not optimal.

Step 5. The only negative entry in C_j-row is $\frac{7}{8}-\frac{M}{4}$ which lies in s_2-column. Therefore the incoming variable is s_2. The ratio 12 is minimum in A_1-row, therefore, the outgoing basic variable is A_1. In further simplex tables, we will not compute the A_1-column. Key element is $\frac{1}{4}$.

New key row is

$$0 \quad s_2 \quad 12 \quad 0 \quad 0 \quad -\frac{3}{2} \quad 1$$

Transformation of R_1. Key column entry in R_1 is $\frac{1}{8}$.

$$\therefore \quad R_1(\text{new}) = R_1(\text{old}) - \frac{1}{8}\,R_2(\text{new})$$

$$\frac{5}{2}-\frac{1}{8}(12)=1$$

$$0-\frac{1}{8}(0)=0$$

$$1-\frac{1}{8}(0)=1$$

$$\frac{5}{16}-\frac{1}{8}\left(-\frac{3}{2}\right)=\frac{1}{2}$$

$$\frac{1}{8}-\frac{1}{8}(1)=0$$

Transformation of R_3. Key column entry in R_3 is $-\frac{1}{4}$

$$\therefore \quad R_3(\text{new}) = R_3(\text{old}) + \frac{1}{4}\,R_2(\text{new})$$

$$1+\frac{1}{4}(12)=4$$

$$1+\frac{1}{4}(0)=1$$

$$0+\frac{1}{4}(0)=0$$

$$-\frac{1}{8}+\frac{1}{4}\left(-\frac{3}{2}\right)=-\frac{1}{2}$$

$$-\frac{1}{4}+\frac{1}{4}(1)=0$$

The new simplex table is given below.

Simplex Table IV

		$c_j \rightarrow$	5	3	0	0	Ratio
c_B	Basis	Solution $b(=x_B)$	x_1	x_2	s_1	s_2	
3	x_2	1	0	1	$\frac{1}{2}$	0	
0	s_2	12	0	0	$-\frac{3}{2}$	1	
5	x_1	4	1	0	$-\frac{1}{2}$	0	
Z = 23		Z_j	5	3	−1	0	
		$C_j = c_j - Z_j$	0	0	1	0	

Since all entries in C_j-row are zero or positive, the current solution is optimal with $x_1 = 4, x_2 = 1, s_1 = 0, s_2 = 12$ and Min. Z = 23.

Example 2. *Use penalty (Big-M) method to solve the following LP problem.*

Maximize $Z = x_1 + 3x_2 - 2x_3$

subject to $-x_1 - 2x_2 - 2x_3 = -6$

$-x_1 - x_2 + x_3 \le -2$

$x_1, x_2, x_3 \ge 0.$

Sol. Step 1. The constants on the right hand side of each constraint are negative. Therefore, multiplying both sides of each constraint by − 1, we get

$$x_1 + 2x_2 + 2x_3 = 6$$
$$x_1 + x_2 - x_3 \ge 2$$

Introducing a surplus variable s and two artificial variables A_1 and A_2, we get the standard form of the LP problem as:

Maximize $Z = x_1 + 3x_2 - 2x_3 + 0s - MA_1 - MA_2$

subject to $x_1 + 2x_2 + 2x_3 + A_1 = 6$

$x_1 + x_2 - x_3 - s + A_2 = 2$

where $x_1, x_2, x_3, s, A_1, A_2 \ge 0$

and M is a large positive number.

Step 2. Since we have 2 equations in 6 variables, a solution is obtained by setting 6 − 2 = 4 variables equal to zero and solving for the remaining 2 variables. By setting $x_1 = x_2 = x_3 = s = 0$, the initial basic feasible solution is $A_1 = 6, A_2 = 2$ and $Z = -8$ M.

The initial basic feasible solution is given in the simplex table below.

Simplex Table I

		$c_j \rightarrow$	1	3	−2	0	−M	−M	Ratio
c_B	Basis	Solution $b(=x_B)$	x_1	x_2	x_3	s	A_1	A_2	x_B/x_2
−M	A_1	6	1	2	2	0	1	0	6/2 = 3
−M	A_2	2	1	1	−1	−1	0	1	2/1 = 2→
Z = − 8 M		Z_j	− 2M	− 3M	− M	M	− M	− M	
		$C_j = c_j - Z_j$	1 + 2M	3 + 3M ↑	− 2 + M	− M	0	0	

Some entries in C_j-row being positive, the current solution is not optimal.

Step 3. Largest positive entry in C_j-row is 3 + 3M which lies in x_2-column. Therefore, the incoming variable is x_2. The ratio 2 is minimum in A_2-row, therefore, the outgoing basic variable is A_2. In further simplex tables, we will not compute the A_2-column. Key element is 1.

New key row is

3 x_2 2 1 1 −1 −1 0

Transformation of R_1. Key column entry in R_1 is 2.

$\therefore$ $R_1(\text{new}) = R_1(\text{old}) - 2R_2(\text{new})$

$6 - 2(2) = 2$
$1 - 2(1) = -1$
$2 - 2(1) = 0$
$2 - 2(-1) = 4$
$0 - 2(-1) = 2$
$1 - 2(0) = 1$

New simplex table is given below.

Simplex Table II

		$c_j \rightarrow$	1	3	−2	0	−M	Ratio
c_B	Basis	Solution $b(=x_B)$	x_1	x_2	x_3	s	A_1	x_B/x_3
− M	A_1	2	−1	0	4	2	1	$\frac{2}{4} = \frac{1}{2}$ →
3	x_2	2	1	1	−1	−1	0	—
Z = 6 − 2 M		Z_j	3 + M	3	− 3 − 4 M	− 3 − 2 M	− M	
		$C_j = c_j - Z_j$	− 2 − M	0	1 + 4 M ↑	3 + 2 M	0	

Some entries in C_j-row being positive, the current solution is not optimal.

Step 4. Largest positive entry in C_j -row is 1 + 4M which lies in x_3- column. Therefore, the incoming variable is x_3. The ratio $\frac{1}{2}$ is minimum in A_1 - row, therefore, the outgoing basic variable is A_1. In further simplex tables, we will not compute the A_1-column. Key element is 4.

New key row is

$$-2 \quad x_3 \quad \frac{1}{2} \quad -\frac{1}{4} \quad 0 \quad 1 \quad \frac{1}{2}$$

Transformation of R_2. Key column entry in R_2 is – 1.

$\therefore$ $R_2(\text{new}) = R_2(\text{old}) + 1\, R_1(\text{new})$

$$2 + 1(\tfrac{1}{2}) = \frac{5}{2}$$
$$1 + 1(-\tfrac{1}{4}) = \frac{3}{4}$$
$$1 + 1(0) = 1$$
$$-1 + 1(1) = 0$$
$$-1 + 1(\tfrac{1}{2}) = -\frac{1}{2}$$

New simplex table is given below.

Simplex Table III

c_B	Basis	$c_j \rightarrow$ Solution $b(= x_B)$	1	3	– 2	0	Ratio
			x_1	x_2	x_3	s	x_B/s
– 2	x_3	$\frac{1}{2}$	$-\frac{1}{4}$	0	1	$\frac{1}{2}$	1 →
3	x_2	$\frac{5}{2}$	$\frac{3}{4}$	1	0	$-\frac{1}{2}$	—
$Z = \frac{13}{2}$		Z_i	$\frac{11}{4}$	3	– 2	$-\frac{5}{2}$	
		$C_j = c_j - Z_j$	$-\frac{7}{4}$	0	0	$\frac{5}{2}$ ↑	

One entry in C_j-row is positive, the current solution is not optimal.

Step 5. Largest positive entry in C_j-row is $\frac{5}{2}$ which lies in s-column. Therefore, the incoming variable is s. Outgoing basic variable is x_3. Key element is $\frac{1}{2}$.

New key row is

$$0 \quad s \quad 1 \quad -\frac{1}{2} \quad 0 \quad 2 \quad 1$$

Transformation of R_2. Key column entry in R_2 is $-\frac{1}{2}$.

$\therefore$ $R_2\,(\text{new}) = R_2\,(\text{old}) + \frac{1}{2}\, R_1(\text{new})$

$$\frac{5}{2} + \frac{1}{2}(1) = 3$$
$$\frac{3}{4} + \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{2}$$
$$1 + \frac{1}{2}(0) = 1$$
$$0 + \frac{1}{2}(2) = 1$$
$$-\frac{1}{2} + \frac{1}{2}(1) = 0$$

New simplex table is given below.

Simplex Table IV

c_B	Basis	$c_j \rightarrow$ Solution $b(= x_B)$	1	3	– 2	0	Ratio
			x_1	x_2	x_3	s	
0	s	1	$-\frac{1}{2}$	0	2	1	
3	x_2	3	$\frac{1}{2}$	1	1	0	
Z = 9		Z_j	$\frac{3}{2}$	3	3	0	
		$C_j = c_j - Z_j$	$-\frac{1}{2}$	0	– 5	0	

Since all entries in C_j-row are negative or zero, the optimal solution has been arrived at with $x_1 = 0, x_2 = 3, x_3 = 0$ and Max. Z = 9.

EXERCISE 7.6

Solve the following LP problems using Big-M method.

1. Minimize $Z = 8x + 12y$
   subject to
   $$2x + 2y \geq 1$$
   $$x + 3y \geq 2$$
   $$x, y \geq 0$$
2. Minimize $Z = 5x + 8y$
   subject to
   $$x + y = 5$$
   $$x \leq 4$$
   $$y \geq 2$$
   $$x, y \geq 0.$$

3. Minimize $Z = 2x_1 + x_2$
subject to
$$3x_1 + x_2 = 3$$
$$4x_1 + 3x_2 \geq 6$$
$$x_1 + 2x_2 \leq 3$$
$$x_1, x_2 \geq 0$$

4. Minimize $z = 3x + 6y$
subject to
$$-x + y \leq 6$$
$$x + y \geq 10$$
$$x, y \geq 0$$

5. Maximize $Z = 2x + y$
subject to
$$x + y \leq 6$$
$$-x + y \geq 4$$
$$x, y \geq 0$$

6. Maximize $Z = -3x + 2y$
subject to
$$x - y \leq 4$$
$$-x + y = 4$$
$$x \geq 6$$
$$x, y \geq 0$$

7. Maximize $Z = 3x + 2y$
subject to
$$-x + y \geq 3$$
$$x + 2y \leq 2$$
$$x, y \geq 0.$$

8. Maximize $Z = 3x + 4y$
subject to
$$x + y \leq 12$$
$$5x + 2y \geq 36$$
$$7x + 4y \geq 14$$
$$x, y \geq 0.$$

9. Minimize $Z = 5x_1 + 4x_2 + 3x_3$
subject to
$$x_1 + x_2 + x_3 \geq 100$$
$$2x_1 + x_2 \geq 50$$
$$x_1, x_2, x_3 \geq 0.$$

10. Minimize $Z = x_1 - x_2 - 3x_3$
subject to
$$x_1 + 2x_2 + x_3 = 4$$
$$x_2 + x_3 = 1$$
$$x_1 + x_2 \leq 6$$
$$x_1, x_2, x_3 \geq 0.$$

11. Minimize $Z = x_1 + 2x_2 + x_3$
subject to
$$x_1 - x_2 - x_3 \leq -1$$
$$6x_1 + 3x_2 + 2x_3 = 12$$
$$x_1, x_2, x_3 \geq 0$$

12. Maximize $Z = 2x_1 + x_2 - x_3$
subject to
$$x_1 + 2x_2 + x_3 \leq 5$$
$$-x_1 + x_2 + x_3 \geq 1$$
$$x_1, x_2, x_3 \geq 0$$

13. Maximize $Z = 3x_1 - 2x_2 + x_3$
subject to
$$x_1 + x_2 + x_3 \leq 1$$
$$x_1 - x_2 + x_3 \geq 2$$
$$x_1 - x_2 - x_3 \leq -6$$
$$x_1, x_2, x_3 \geq 0.$$

14. Maximize $Z = 3x_1 + 2x_2 + 3x_3$
subject to
$$2x_1 + x_2 + x_3 \leq 2$$
$$3x_1 + 4x_2 + 2x_3 \geq 8$$
$$x_1, x_2, x_3 \geq 0.$$

15. Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$
subject to
$$x_1 + 2x_2 + 3x_3 = 15$$
$$2x_1 + x_2 + 5x_3 = 20$$
$$x_1 + 2x_2 + x_3 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0.$$

Answers

1. $x = 0, y = \frac{2}{3}$; Min. $Z = 8$
2. $x = 3, y = 2$; Min. $Z = 31$
3. $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$; Min. $Z = \frac{12}{5}$
4. $x = 10, y = 0$; Min. $Z = 30$
5. $x = 1, y = 5$; Max. $Z = 7$
6. $x = 6, y = 10$; Max. $Z = 2$
7. No optimal solution (since in the simplex table II, all entries in C_j-row are negative or zero so that the simplex procedure terminates. But an artificial variable with non-zero value exists in the basis).
8. $x = 4, y = 8$; Max. $Z = 44$
9. $x_1 = 25, x_2 = 0, x_3 = 75$; Min. $Z = 350$
10. $x_1 = 3, x_2 = 0, x_3 = 1$; Min. $Z = 0$
11. $x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{9}{4}$; Min. $Z = \frac{7}{2}$
12. $x_1 = 1, x_2 = 2, x_3 = 0$; Max. $Z = 4$
13. No optimal solution
14. $x_1 = 0, x_2 = 2, x_3 = 0$; Max. $Z = 4$
15. $x_1 = x_2 = x_3 = \frac{5}{2}, x_4 = 0$; Max. $Z = 15$.

7.15. DUALITY CONCEPT

With every linear programming problem, there exists another linear programming problem called its dual since every LPP can be analysed in two different ways without any additional data or information. For example, profit maximization problem can be seen as a problem of cost minimization and cost minimization problem can be viewed as a problem of maximizing the efficiency of using available resources. The original problem is called **primal** and the associated problem is called its **dual**. In general, either problem can be considered as primal and the other its dual.

7.16. FORMULATION OF A DUAL PROBLEM

Suppose the primal LP problem is in the form

Maximize $Z = c_1x_1 + c_2x_2 + \dots\dots\dots + c_nx_n$

subject to the constraints
$$a_{11}x_1 + a_{12}x_2 + \dots\dots\dots + a_{1n}x_n \leq b_1$$
$$a_{21}x_1 + a_{22}x_2 + \dots\dots\dots + a_{2n}x_n \leq b_2$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots\dots\dots$$
$$a_{m1}x_1 + a_{m2}x_2 + \dots\dots\dots + a_{mn}x_n \leq b_n$$
and
$$x_1, x_2, \dots\dots\dots, x_n \geq 0$$

The corresponding dual problem is

Minimize $Z^* = b_1 y_1 + b_2 y_2 + \ldots\ldots\ldots + b_m y_m$

subject to the constraints

$$a_{11} y_1 + a_{21} y_2 + \ldots\ldots\ldots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \ldots\ldots\ldots + a_{m2} y_m \geq c_2$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots$$
$$a_{1n} y_1 + a_{2n} y_2 + \ldots\ldots\ldots + a_{mn} y_m \geq c_n$$

and $$y_1, y_2, \ldots\ldots, y_m \geq 0$$

The characteristics of the primal and the dual problem can be stated as follows:

(i) The maximization problem in the primal becomes the minimization problem in the dual and *vice versa*.

(ii) The $\leq$ type constraints in the primal become $\geq$ type constraints in the dual and *vice versa*.

(iii) The co-efficients $c_1, c_2, \ldots\ldots, c_n$ in the objective function of the primal become the constants $b_1, b_2, \ldots\ldots, b_m$ in the constraints of the dual and *vice versa*.

(iv) If the primal has n variables and m constraints, then the dual has m variables and n constraints. Thus the body matrix of dual is the transpose of body matrix of primal and *vice versa*.

(v) A new set of variables appear in the dual.

(vi) The variables in both the primal and the dual are non-negative.

(vii) The dual of the dual problem is the original primal problem itself.

ILLUSTRATIVE EXAMPLES

Examples 1. *Obtain the dual of*

Maximize $Z = 5x_1 + 3x_2$

subject to
$$x_1 + x_2 \leq 2$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

(M.D.U. Dec. 2011)

Sol. (i) Primal is of maximization type

Dual will be of 'minimization' type.

(ii) Primal has 2 variables x_1, x_2 and 3 constraints.

Dual will have 3 variables y_1, y_2, y_3 and 2 constraints.

(iii) Primal has $\leq$ constraints.

Dual will have $\geq$ constraints.

(iv) Primal has $c_1 = 5$, $c_2 = 3$ and $b_1 = 2$, $b_2 = 10$, $b_3 = 12$

Dual will have $c_1 = 2$, $c_2 = 10$, $c_3 = 12$ and $b_1 = 5$, $b_2 = 3$

The body matrix in the primal is $\begin{bmatrix} 1 & 1 \\ 5 & 2 \\ 3 & 8 \end{bmatrix}$

The body matrix in the dual will be

$$\begin{bmatrix} 1 & 1 \\ 5 & 2 \\ 3 & 8 \end{bmatrix}' = \begin{bmatrix} 1 & 5 & 3 \\ 1 & 2 & 8 \end{bmatrix}$$

Thus the dual is

Minimize $Z^* = 2y_1 + 10y_2 + 12y_3$

subject to
$$y_1 + 5y_2 + 3y_3 \geq 5$$
$$y_1 + 2y_2 + 8y_3 \geq 3$$

and $$y_1, y_2, y_3 \geq 0$$

7.17. DUALITY PRINCIPLE

If either the primal or the dual problem has an optimal solution then so does the other problem and the optimal value of the primal's objective function is the same as that of its dual.

Example 2. *Solve the dual problem of the following LPP and hence find Max. Z:*

Maximize $Z = 20x_1 + 30x_2$

subject to
$$3x_1 + 3x_2 \leq 36$$
$$5x_1 + 2x_2 \leq 50$$
$$2x_1 + 6x_2 \leq 60$$
$$x_1, x_2 \geq 0.$$

Sol. The dual of the above LPP is:

Minimize $Z^* = 36y_1 + 50y_2 + 60y_3$

subject to
$$3y_1 + 5y_2 + 2y_3 \geq 20$$
$$3y_1 + 2y_2 + 5y_3 \geq 30$$
$$y_1, y_2, y_3 \geq 0$$

Solving this problem by simplex method, it can be verified that the final simplex table is:

		$c_j \rightarrow$	36	50	60	0	0	M	M
c_B	Basis	Solution $b\ (= y_B)$	y_1	y_2	y_3	s_1	s_2	A_1	A_2
36	y_1	5	1	$\frac{13}{6}$	0	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$
60	y_3	$\frac{5}{2}$	0	$-\frac{3}{4}$	1	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$
$Z^* = 330$		Z_j^*	36	33	60	−3	−9	3	9
		$C_j = c_j - Z_j^*$	0	17	0	3	9	M − 3	M − 9

The optimal solution to the dual problem is

$$y_1 = 5, y_2 = 0, y_3 = \frac{5}{2} \quad \text{and Min. } Z^* = 330$$

By duality principle = Max. Z = Min. Z* = 330.

7.18. DUAL SIMPLEX METHOD

Dual simplex method is similar to the regular simplex method which has already been discussed. The only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable whereas in the regular simplex method, the reverse is done. The regular simplex method starts with a basic feasible but non-optimal solution and works towards optimality, the dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility.

7.19. WORKING PROCEDURE FOR DUAL SIMPLEX METHOD

Setp 1: Check whether the objective function is to be maximized or minimized. If the objective function is to be minimized, then convert it into maximization form.

Setp 2: Convert $\geq$ type constraints, if any, into $\leq$ type by multiplying such constraints by -1.

Step 3: Express all constraints as equations by adding slack variables, one for each constraint.

Step 4: Find the initial basic solution and express this information in the form of a table as in regular simplex method.

Step 5: Compute $C_j = c_j - Z_j$

(*a*) If all $C_j \leq 0$ and all $b_i \geq 0$, then the solution found above is the optimal basic feasible solution.

(*b*) If all $C_j \leq 0$ and at least one $b_i < 0$, then go to the next step.

(*c*) If any $C_j > 0$, the method fails.

Step 6: *Selection of key row and the outgoing variable*

Select the row containing the most negative b_i. This row is the key row and the basic variable heading the key row is the outgoing variable.

Step 7: *Selection of key column and the incoming variable*

(*a*) If all elements in the key row are ≥ 0, then the problem does not have feasible solution.

(*b*) If at least one element in the key row is negative, find the ratios of the corresponding elements of C_j-row to these elements. Ignore the ratios associated with positive or zero elements of the key row. Choose the smallest of these ratios. The corresponding column is the key column and the variable heading the key column is the incoming variable.

Step 8: Mark the key element and make it unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Using dual simplex method*

Maximize $Z = -3x_1 - x_2$

subject to $x_1 + x_2 \geq 1,\ 2x_1 + 3x_2 \geq 2,\ x_1, x_2 \geq 0.$

(M.D.U. Dec. 2008, Dec., 2009;K.U.K. Dec. 2009, Dec. 2010)

Sol.

Step 1. The problem is that of maximization.

Step 2. Converting $\geq$ type constraints into $\leq$ type, the L.P.P. takes the form

Maximize $Z = -3x_1 - x_2$

subject to $-x_1 - x_2 \leq -1,\ -2x_1 - 3x_2 \leq -2,\ x_1, x_2 \geq 0$

Step 3. Introducing slack variables s_1, s_2 (one for each constraint), the problem in standard form is:

Maximize $\quad Z = -3x_1 - x_2 + 0s_1 + 0s_2$

subject to $\quad -x_1 - x_2 + s_1 = -1$

$-2x_1 - 3x_2 + s_2 = -2$

$x_1, x_2, s_1, s_2 \geq 0$

Step 4. The initial basic solution is given by

$$x_1 = x_2 = 0,\ S_1 = -1,\ S_2 = -2 \text{ at which } Z = 0$$

Dual Simplex Table I

C_B	Basis	$C_j \to$ Solution $b\,(= x_B)$	-3 x_1	-1 x_2	0 s_1	0 s_2
0	S_1	-1	-1	-1	1	0
0	S_2	-2	-2	-3	0	1
$Z = 0$		Z_j	0	0	0	0
		$C_j = c_j - Z_j$	-3	-1	0	0

Step 5. Since all $C_j \leq 0$ and all $b_i < 0$, the initial solution is optimal but infeasible. So we proceed further.

Step 6. The most negative b_i (*i.e.*, the b_i which is negative and numerically largest) is $b_2 = -2$, **the second row is the key row and S_2 is the outgoing variable.**

Step 7. The key row has negative entries. The ratios of the elements in C_j-row to the corresponding negative elements of the key row (neglecting ratios corresponding to positive or zero elements of key row) are

$$\frac{-3}{-2} = \frac{3}{2},\ \frac{-1}{-3} = \frac{1}{3}$$

Smallest ratio is $\frac{1}{3}$. **The x_2-column is the key column and x_2 is the incoming variable.**

Step 8. The key element is -3, shown circled. Drop the outgoing variable s_2 from the basis and introduce the incoming variable x_2. The new basis contains s_1, x_2 as the basic variables. The co-efficient of x_2 in the objective function is -1. Therefore, the entry in C_B column corresponding to the new basic variable x_2 will be -1. Since the key element enclosed in the circle is not 1, divide all elements of the key row by the key element -3 to make it unity and make all other elements in the key column zero. Thus the key row

$$\begin{array}{ccccccc} 0 & s_2 & -2 & -2 & -3 & 0 & 1 \end{array}$$

is replaced by the new key row

$$\begin{array}{ccccccc} -1 & x_2 & \frac{2}{3} & \frac{2}{3} & 1 & 0 & -\frac{1}{3} \end{array}$$

Now we make all other elements of the key column (*i.e.*, x_2-column) zero.

Transformation of R_1. Key column entry in R_1 is -1

$$\therefore \quad R_1\text{ (new)} = R_1\text{ (old)} - (-1)\, R_2\text{ (new)}$$

$$-1 + 1\left(\frac{2}{3}\right) = -\frac{1}{3}$$

$$-1 + 1\left(\frac{2}{3}\right) = -\frac{1}{3}$$

$$-1 + 1\,(1) = 0$$

$$1 + 1\,(0) = 1$$

$$0 + 1\left(-\frac{1}{3}\right) = -\frac{1}{3}$$

The above information is summarized in the following table:

Dual Simplex Table II

C_B	Basis	c_j Solution $b\ (= x_B)$	-3	-1	0	0
			x_1	x_2	s_1	s_2
0	s_1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	1	$-\frac{1}{3}$
-1	x_2	$\frac{2}{3}$	$\frac{2}{3}$	1	0	$-\frac{1}{3}$
$Z = -\frac{2}{3}$		Z_j	$-\frac{2}{3}$	-1	0	$\frac{1}{3}$
		$C_j = c_j - Z_j$	$-\frac{7}{3}$	0	0	$-\frac{1}{3}$

Step 9. Test for optimality. Here all $C_j \le 0$ but $b_1 = -\frac{1}{3} < 0$, the current solution is not optimal. Therefore, an improvement in the value of Z is possible and we repeat steps 6 to 8.

The first row is the key row and s_1 is the outgoing variable. The ratios of the elements in C_j-row to the corresponding negative elements of the key row are

$$\frac{-\frac{7}{3}}{-\frac{1}{3}} = 7,\ \frac{-\frac{1}{3}}{-\frac{1}{3}} = 1$$

The smallest ratio is 1, therefore, s_2-column is the key column and s_2 is the incoming variable. The key element is $-\frac{1}{3}$, shown circled. Drop s_1 and introduce s_2. The new basis contains s_2, x_2 as the basic variables. Dividing all elements of the key row by the key element $-\frac{1}{3}$, the new key row is

$$\begin{array}{ccccccc} 0 & s_2 & 1 & 1 & 0 & -3 & 1 \end{array}$$

Transformation of R_2. Key column entry in R_2 is $-\frac{1}{3}$.

$$\therefore \quad R_2\text{ (new)} = R_2\text{ (old)} - \left(-\frac{1}{3}\right) R_1\text{ (new)}$$

$$\frac{2}{3} + \frac{1}{3}(1) = 1$$

$$\frac{2}{3} + \frac{1}{3}(1) = 1$$

$$1 + \frac{1}{3}(0) = 1$$

$$0 + \frac{1}{3}(-3) = -1$$

$$-\frac{1}{3} + \frac{1}{3}(1) = 0$$

The above information is summarized in the following table:

Dual Simplex Table III

C_B	Basis	C_j Solution $b\ (= x_B)$	-3	-1	0	0
			x_1	x_2	s_1	s_2
0	s_2	1	1	0	-3	1
-1	x_2	1	1	1	-1	0
$Z = -1$		Z_j	-1	-1	1	0
		$C_j = c_j - Z_j$	-2	0	-1	0

Here all $C_j \le 0$ and all $b_i > 0$, the optimal feasible solution has been attained. Thus the optimal feasible solution is

$$x_1 = 0,\ x_2 = 1,\ \text{Max. } Z = -1.$$

Example 2. *Using dual simplex method,*

Minimize $Z = x_1 + 2x_2 + 3x_3$

subject to $2x_1 - x_2 + x_3 \ge 4$

$$x_1 + x_2 + 2x_3 \le 8$$

$$x_2 - x_3 \ge 2$$

$$x_1, x_2, x_3 \ge 0.$$

Sol.

Step 1. The problem is that of minimization. Converting it into maximization problem by using the relationship

Min. $Z = -$ Max. Z^*, where $Z^* = -Z$, we have

Maximize $\quad Z^* = -x_1 - 2x_2 - 3x_3$

subject to $\quad 2x_1 - x_2 + x_3 \geq 4$

$$x_1 + x_2 + 2x_3 \leq 8$$
$$x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

Step 2. Converting $\geq$ type constraints into $\leq$ type, the L.P.P. takes the form

Maximize $Z^* = -x_1 - 2x_2 - 3x_3$

subject to $-2x_1 + x_2 - x_3 \leq -4$

$$x_1 + x_2 + 2x_3 \leq 8$$
$$-x_2 + x_3 \leq -2$$
$$x_1, x_2, x_3 \geq 0$$

Step 3. Introducing slack variables s_1, s_2, s_3, the problem in standard form is:

Maximize $Z^* = -x_1 - 2x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$

subject to $\quad -2x_1 + x_2 - x_3 + s_1 = -4$

$$x_1 + x_2 + 2x_3 + s_2 = 8$$
$$-x_2 + x_3 + s_3 = -2$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Step 4. The initial basic solution is given by

$$x_1 = x_2 = x_3 = 0,\ s_1 = -4,\ s_2 = 8,\ s_3 = -2$$

at which $Z^* = 0$

Dual Simplex Table I

C_B	Basis	$C_j \rightarrow$ Solution $b\ (= x_B)$	-1	-2	-3	0	0	0
			x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	-4	-2	1	-1	1	0	0
0	s_2	8	1	1	2	0	1	0
0	s_3	-2	0	-1	1	0	0	1
$Z^* = 0$		Z_j^*	0	0	0	0	0	0
		$C_j = c_j - Z_j^*$	-1	-2	-3	0	0	0

Step 5. Since all $C_j \leq 0$ and at least one $b_i < 0$, the initial solution is optimal but infeasible. So we proceed further.

Step 6. The most negative b_i is $b_1 = -4$, the first row is the key row and s_1 is the outgoing variable.

Step 7. The key row has negative entries. The ratios of the elements in C_j-row to the corresponding negative elements of the key row are

$$\frac{-1}{-2} = \frac{1}{2},\ \frac{-3}{-1} = 3$$

Smallest ratio is $\frac{1}{2}$. The x_1-column is the key column and x_1 is the incoming variable.

Step 8. The key element is -2, shown circled. The new basis contains x_1, s_2, s_3 as the basic variables. Since the key element is not 1, divide all elements of the key row by the key element -2 to make it unity. Thus, the new key-row is

$$-1 \quad x_1 \quad 2 \quad 1 \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

Now we make all other elements of the key column zero.

Transformation of R_2. Key column entry in R_2 is 1.

$\therefore \quad R_2\ (\text{new}) = R_2\ (\text{old}) - 1\ R_1\ (\text{new})$

$$8 - 1(2) = 6,\quad 1 - 1(1) = 0,\quad 1 - 1\left(-\frac{1}{2}\right) = \frac{3}{2}$$
$$2 - 1\left(\frac{1}{2}\right) = \frac{3}{2},\quad 0 - 1\left(-\frac{1}{2}\right) = \frac{1}{2},\quad 1 - 1(0) = 1$$
$$0 - 1(0) = 0$$

Transformation of R_3. Key column entry in R_3 is 0.

$\therefore \quad R_3\ (\text{new}) = R_3\ (\text{old})$

The above information is summarized in the following table:

Dual Simplex Table II

C_B	Basis	$C_j \rightarrow$ Solution $b\ (= x_B)$	-1	-2	-3	0	0	0
			x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	2	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
0	s_2	6	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	1	0
0	s_3	-2	0	-1	1	0	0	1
$Z^* = -2$		Z_j^*	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
		$C_j = c_j - Z_j^*$	0	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	0	0

Step 9. Here all $C_j \leq 0$ but $b_3 = -2 < 0$, the current solution is not optimal. So we repeat steps 6 to 8.

The third row is the key row and s_3 is the outgoing variable. The ratios of the elements in C_j-row to the corresponding negative elements of the key row are

$$\frac{-\frac{5}{2}}{-1} = \frac{5}{2}$$

The x_2-column is the key column and x_2 is the incoming variable. The key element is -1, shown circled. Drop s_3 and introduce x_2. The new basis contains x_1, s_2, x_2 as the basic variables. Dividing all elements of the key row by the key element -1, the new key row is

$$-2 \quad x_2 \quad 2 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad -1$$

Now we make all other elements of the key column (*i.e.*, x_2-column) zero.

Transformation of R_1. Key column entry in R_1 is $-\frac{1}{2}$.

$$\therefore \quad R_1\text{ (new)} = R_1\text{ (old)} - \left(-\frac{1}{2}\right) R_3\text{ (new)}$$

$$2 + \frac{1}{2}(2) = 3, \quad 1 + \frac{1}{2}(0) = 1, \quad -\frac{1}{2} + \frac{1}{2}(1) = 0$$

$$\frac{1}{2} + \frac{1}{2}(-1) = 0, \quad -\frac{1}{2} + \frac{1}{2}(0) = -\frac{1}{2}, \quad 0 + \frac{1}{2}(0) = 0$$

$$0 + \frac{1}{2}(-1) = -\frac{1}{2}$$

Transformation of R_2. Key column entry in R_2 is $\frac{3}{2}$.

$$\therefore \quad R_2\text{ (new)} = R_2\text{ (old)} - \frac{3}{2} R_3\text{ (new)}$$

$$6 - \frac{3}{2}(2) = 3, \quad 0 - \frac{3}{2}(0) = 0, \quad \frac{3}{2} - \frac{3}{2}(1) = 0$$

$$\frac{3}{2} - \frac{3}{2}(-1) = 3, \quad \frac{1}{2} - \frac{3}{2}(0) = \frac{1}{2}, \quad 1 - \frac{3}{2}(0) = 1$$

$$0 - \frac{3}{2}(-1) = \frac{3}{2}$$

The above information is summarized in the following table:

Dual Simplex Table III

C_B	Basis	$C_j \rightarrow$	-1	-2	-3	0	0	0
		Solution $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	3	1	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
0	s_2	3	0	0	3	$\frac{1}{2}$	1	$\frac{3}{2}$
-2	x_2	2	0	1	-1	0	0	-1
$Z^* = -7$		Z_j^*	-1	-2	2	$\frac{1}{2}$	0	$\frac{5}{2}$
		$C_j = c_j - Z_j^*$	0	0	-5	$-\frac{1}{2}$	0	$-\frac{5}{2}$

Step 5. The only negative entry in C_j-row is $\frac{7}{8} - \frac{M}{4}$ which lies in s_2-column. Therefore the incoming variable is s_2. The ratio 12 is minimum in A_1-row, therefore, the outgoing basic variable is A_1. In further simplex tables, we will not compute the A_1-column. Key element is $\frac{1}{4}$.

New key row is

$$0 \quad s_2 \quad 12 \quad 0 \quad 0 \quad -\frac{3}{2} \quad 1$$

Transformation of R_1. Key column entry in R_1 is $\frac{1}{8}$.

$$\therefore \quad R_1(\text{new}) = R_1(\text{old}) - \frac{1}{8} R_2(\text{new})$$

$$\frac{5}{2} - \frac{1}{8}(12) = 1$$

$$0 - \frac{1}{8}(0) = 0$$

$$1 - \frac{1}{8}(0) = 1$$

$$\frac{5}{16} - \frac{1}{8}\left(-\frac{3}{2}\right) = \frac{1}{2}$$

$$\frac{1}{8} - \frac{1}{8}(1) = 0$$

Transformation of R_3. Key column entry in R_3 is $-\frac{1}{4}$

$$\therefore \quad R_3(\text{new}) = R_3(\text{old}) + \frac{1}{4} R_2(\text{new})$$

$$1 + \frac{1}{4}(12) = 4$$

$$1 + \frac{1}{4}(0) = 1$$

$$0 + \frac{1}{4}(0) = 0$$

$$-\frac{1}{8} + \frac{1}{4}\left(-\frac{3}{2}\right) = -\frac{1}{2}$$

$$-\frac{1}{4} + \frac{1}{4}(1) = 0$$

The new simplex table is given below.

Simplex Table IV

c_B	Basis	$c_j \rightarrow$ Solution $b(= x_B)$	5 x_1	3 x_2	0 s_1	0 s_2	Ratio
3	x_2	1	0	1	$\frac{1}{2}$	0	
0	s_2	12	0	0	$-\frac{3}{2}$	1	
5	x_1	4	1	0	$-\frac{1}{2}$	0	
Z = 23		Z_j	5	3	– 1	0	
		$C_j = c_j - Z_j$	0	0	1	0	

Since all entries in C_j-row are zero or positive, the current solution is optimal with $x_1 = 4$, $x_2 = 1$, $s_1 = 0$, $s_2 = 12$ and Min. Z = 23.

Example 2. *Use penalty (Big-M) method to solve the following LP problem.*

Maximize $Z = x_1 + 3x_2 - 2x_3$

subject to
$$-x_1 - 2x_2 - 2x_3 = -6$$
$$-x_1 - x_2 + x_3 \leq -2$$
$$x_1, x_2, x_3 \geq 0.$$

Sol. Step 1. The constants on the right hand side of each constraint are negative. Therefore, multiplying both sides of each constraint by – 1, we get

$$x_1 + 2x_2 + 2x_3 = 6$$
$$x_1 + x_2 - x_3 \geq 2$$

Introducing a surplus variable s and two artificial variables A_1 and A_2, we get the standard form of the LP problem as:

Maximize $Z = x_1 + 3x_2 - 2x_3 + 0s - MA_1 - MA_2$

subject to
$$x_1 + 2x_2 + 2x_3 + A_1 = 6$$
$$x_1 + x_2 - x_3 - s + A_2 = 2$$

where $x_1, x_2, x_3, s, A_1, A_2 \geq 0$

and M is a large positive number.

Step 2. Since we have 2 equations in 6 variables, a solution is obtained by setting 6 – 2 = 4 variables equal to zero and solving for the remaining 2 variables. By setting $x_1 = x_2 = x_3 = s = 0$, the initial basic feasible solution is $A_1 = 6$, $A_2 = 2$ and Z = – 8 M.

The initial basic feasible solution is given in the simplex table below.

Simplex Table I

c_B	Basis	$c_j \rightarrow$ Solution $b(= x_B)$	1 x_1	3 x_2	– 2 x_3	0 s	– M A_1	– M A_2	Ratio x_B/x_2
– M	A_1	6	1	2	2	0	1	0	6/2 = 3
– M	A_2	2	1	1	– 1	– 1	0	1	2/1 = 2→
Z = – 8 M		Z_j	– 2M	– 3M	– M	M	– M	– M	
		$C_j = c_j - Z_j$	1 + 2M	3 + 3M ↑	– 2 + M	– M	0	0	

Some entries in C_j-row being positive, the current solution is not optimal.

Step 3. Largest positive entry in C_j-row is 3 + 3M which lies in x_2-column. Therefore, the incoming variable is x_2. The ratio 2 is minimum in A_2-row, therefore, the outgoing basic variable is A_2. In further simplex tables, we will not compute the A_2-column. Key element is 1.

New key row is

3 x_2 2 1 1 – 1 – 1 0

Transformation of R_1. Key column entry in R_1 is 2.

∴

$R_1(\text{new}) = R_1(\text{old}) - 2R_2(\text{new})$
6 – 2(2) = 2
1 – 2(1) = – 1
2 – 2(1) = 0
2 – 2(– 1) = 4
0 – 2(– 1) = 2
1 – 2(0) = 1

New simplex table is given below.

Simplex Table II

c_B	Basis	$c_j \rightarrow$ Solution $b(= x_B)$	1 x_1	3 x_2	– 2 x_3	0 s	– M A_1	Ratio x_B/x_3
– M	A_1	2	– 1	0	4	2	1	$\frac{2}{4} = \frac{1}{2}$ →
3	x_2	2	1	1	–1	–1	0	—
Z = 6 – 2 M		Z_j	3 + M	3	– 3 – 4 M	– 3 – 2 M	– M	
		$C_j = c_j - Z_j$	– 2 – M	0	1 + 4 M ↑	3 + 2 M	0	

Some entries in C_j-row being positive, the current solution is not optimal.

Step 4. Largest positive entry in C_j -row is $1 + 4M$ which lies in x_3- column. Therefore, the incoming variable is x_3. The ratio $\frac{1}{2}$ is minimum in A_1 - row, therefore, the outgoing basic variable is A_1. In further simplex tables, we will not compute the A_1-column. Key element is 4.

New key row is

$$-2 \quad x_3 \quad \frac{1}{2} \quad -\frac{1}{4} \quad 0 \quad 1 \quad \frac{1}{2}$$

Transformation of R_2. Key column entry in R_2 is -1.

$\therefore$ $R_2(\text{new}) = R_2(\text{old}) + 1\ R_1(\text{new})$

$$2 + 1(\tfrac{1}{2}) = \frac{5}{2}$$
$$1 + 1(-\tfrac{1}{4}) = \frac{3}{4}$$
$$1 + 1(0) = 1$$
$$-1 + 1(1) = 0$$
$$-1 + 1(\tfrac{1}{2}) = -\frac{1}{2}$$

New simplex table is given below.

Simplex Table III

		$c_j \rightarrow$	1	3	-2	0	Ratio
c_B	Basis	Solution $b(= x_B)$	x_1	x_2	x_3	s	x_B/s
-2	x_3	$\frac{1}{2}$	$-\frac{1}{4}$	0	1	$\frac{1}{2}$	$1 \rightarrow$
3	x_2	$\frac{5}{2}$	$\frac{3}{4}$	1	0	$-\frac{1}{2}$	—
$Z = \frac{13}{2}$		Z_i	$\frac{11}{4}$	3	-2	$-\frac{5}{2}$	
		$C_j = c_j - Z_j$	$-\frac{7}{4}$	0	0	$\frac{5}{2}$ ↑	

One entry in C_j-row is positive, the current solution is not optimal.

Step 5. Largest positive entry in C_j-row is $\frac{5}{2}$ which lies in s-column. Therefore, the incoming variable is s. Outgoing basic variable is x_3. Key element is $\frac{1}{2}$.

New key row is

$$0 \quad s \quad 1 \quad -\frac{1}{2} \quad 0 \quad 2 \quad 1$$

Transformation of R_2. Key column entry in R_2 is $-\frac{1}{2}$.

$\therefore$ $R_2\ (\text{new}) = R_2\ (\text{old}) + \frac{1}{2}\ R_1(\text{new})$

$$\frac{5}{2} + \frac{1}{2}(1) = 3$$
$$\frac{3}{4} + \frac{1}{2}\left(-\frac{1}{2}\right) = \frac{1}{2}$$
$$1 + \frac{1}{2}(0) = 1$$
$$0 + \frac{1}{2}(2) = 1$$
$$-\frac{1}{2} + \frac{1}{2}(1) = 0$$

New simplex table is given below.

Simplex Table IV

		$c_j \rightarrow$	1	3	-2	0	Ratio
c_B	Basis	Solution $b(= x_B)$	x_1	x_2	x_3	s	
0	s	1	$-\frac{1}{2}$	0	2	1	
3	x_2	3	$\frac{1}{2}$	1	1	0	
$Z = 9$		Z_j	$\frac{3}{2}$	3	3	0	
		$C_j = c_j - Z_j$	$-\frac{1}{2}$	0	-5	0	

Since all entries in C_j-row are negative or zero, the optimal solution has been arrived at with $x_1 = 0, x_2 = 3, x_3 = 0$ and Max. $Z = 9$.

EXERCISE 7.6

Solve the following LP problems using Big-M method.

1. Minimize $Z = 8x + 12y$
   subject to
   $$2x + 2y \geq 1$$
   $$x + 3y \geq 2$$
   $$x, y \geq 0$$
2. Minimize $Z = 5x + 8y$
   subject to
   $$x + y = 5$$
   $$x \leq 4$$
   $$y \geq 2$$
   $$x, y \geq 0.$$

3. Minimize $Z = 2x_1 + x_2$
subject to
$$\begin{aligned} 3x_1 + x_2 &= 3 \\ 4x_1 + 3x_2 &\geq 6 \\ x_1 + 2x_2 &\leq 3 \\ x_1, x_2 &\geq 0 \end{aligned}$$

4. Minimize $z = 3x + 6y$
subject to
$$\begin{aligned} -x + y &\leq 6 \\ x + y &\geq 10 \\ x, y &\geq 0 \end{aligned}$$

5. Maximize $Z = 2x + y$
subject to
$$\begin{aligned} x + y &\leq 6 \\ -x + y &\geq 4 \\ x, y &\geq 0 \end{aligned}$$

6. Maximize $Z = -3x + 2y$
subject to
$$\begin{aligned} x - y &\leq 4 \\ -x + y &= 4 \\ x &\geq 6 \\ x, y &\geq 0 \end{aligned}$$

7. Maximize $Z = 3x + 2y$
subject to
$$\begin{aligned} -x + y &\geq 3 \\ x + 2y &\leq 2 \\ x,\ y &\geq 0. \end{aligned}$$

8. Maximize $Z = 3x + 4y$
subject to
$$\begin{aligned} x + y &\leq 12 \\ 5x + 2y &\geq 36 \\ 7x + 4y &\geq 14 \\ x, y &\geq 0. \end{aligned}$$

9. Minimize $Z = 5x_1 + 4x_2 + 3x_3$
subject to
$$\begin{aligned} x_1 + x_2 + x_3 &\geq 100 \\ 2x_1 + x_2 &\geq 50 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

10. Minimize $Z = x_1 - x_2 - 3x_3$
subject to
$$\begin{aligned} x_1 + 2x_2 + x_3 &= 4 \\ x_2 + x_3 &= 1 \\ x_1 + x_2 &\leq 6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

11. Minimize $Z = x_1 + 2x_2 + x_3$
subject to
$$\begin{aligned} x_1 - x_2 - x_3 &\leq -1 \\ 6x_1 + 3x_2 + 2x_3 &= 12 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

12. Maximize $Z = 2x_1 + x_2 - x_3$
subject to
$$\begin{aligned} x_1 + 2x_2 + x_3 &\leq 5 \\ -x_1 + x_2 + x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

13. Maximize $Z = 3x_1 - 2x_2 + x_3$
subject to
$$\begin{aligned} x_1 + x_2 + x_3 &\leq 1 \\ x_1 - x_2 + x_3 &\geq 2 \\ x_1 - x_2 - x_3 &\leq -6 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

14. Maximize $Z = 3x_1 + 2x_2 + 3x_3$
subject to
$$\begin{aligned} 2x_1 + x_2 + x_3 &\leq 2 \\ 3x_1 + 4x_2 + 2x_3 &\geq 8 \\ x_1, x_2, x_3 &\geq 0. \end{aligned}$$

15. Maximize $Z = x_1 + 2x_2 + 3x_3 - x_4$
subject to
$$\begin{aligned} x_1 + 2x_2 + 3x_3 &= 15 \\ 2x_1 + x_2 + 5x_3 &= 20 \\ x_1 + 2x_2 + x_3 + x_4 &= 10 \\ x_1, x_2, x_3, x_4 &\geq 0. \end{aligned}$$

Answers

1. $x = 0, y = \frac{2}{3}$; Min. $Z = 8$

2. $x = 3, y = 2$; Min. $Z = 31$

3. $x_1 = \frac{3}{5}, x_2 = \frac{6}{5}$; Min. $Z = \frac{12}{5}$

4. $x = 10, y = 0$; Min. $Z = 30$

5. $x = 1, y = 5$; Max. $Z = 7$

6. $x = 6, y = 10$; Max. $Z = 2$

7. No optimal solution (since in the simplex table II, all entries in C_j-row are negative or zero so that the simplex procedure terminates. But an artificial variable with non-zero value exists in the basis).

8. $x = 4, y = 8$; Max. $Z = 44$

9. $x_1 = 25, x_2 = 0, x_3 = 75$; Min. $Z = 350$

10. $x_1 = 3, x_2 = 0, x_3 = 1$; Min. $Z = 0$

11. $x_1 = \frac{5}{4}, x_2 = 0, x_3 = \frac{9}{4}$; Min. $Z = \frac{7}{2}$

12. $x_1 = 1, x_2 = 2, x_3 = 0$; Max. $Z = 4$

13. No optimal solution

14. $x_1 = 0, x_2 = 2, x_3 = 0$; Max. $Z = 4$

15. $x_1 = x_2 = x_3 = \frac{5}{2}, x_4 = 0$; Max. $Z = 15$.

7.15. DUALITY CONCEPT

With every linear programming problem, there exists another linear programming problem called its dual since every LPP can be analysed in two different ways without any additional data or information. For example, profit maximization problem can be seen as a problem of cost minimization and cost minimization problem can be viewed as a problem of maximizing the efficiency of using available resources. The original problem is called **primal** and the associated problem is called its **dual**. In general, either problem can be considered as primal and the other its dual.

7.16. FORMULATION OF A DUAL PROBLEM

Suppose the primal LP problem is in the form

Maximize $Z = c_1x_1 + c_2x_2 + \ldots\ldots\ldots\ldots + c_nx_n$

subject to the constraints
$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots\ldots\ldots\ldots + a_{1n}x_n &\leq b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots\ldots\ldots\ldots + a_{2n}x_n &\leq b_2 \\ &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots\ldots\ldots\ldots + a_{mn}x_n &\leq b_n \end{aligned}$$
and
$$x_1, x_2, \ldots\ldots\ldots\ldots, x_n \geq 0$$

The corresponding dual problem is

Minimize $Z^* = b_1 y_1 + b_2 y_2 + \dots\dots + b_m y_m$

subject to the constraints

$$a_{11} y_1 + a_{21} y_2 + \dots\dots + a_{m1} y_m \geq c_1$$
$$a_{12} y_1 + a_{22} y_2 + \dots\dots + a_{m2} y_m \geq c_2$$
$$\dots\dots\dots\dots\dots\dots$$
$$\dots\dots\dots\dots\dots\dots$$
$$a_{1n} y_1 + a_{2n} y_2 + \dots\dots + a_{mn} y_m \geq c_n$$

and $\quad y_1, y_2, \dots\dots, y_m \geq 0$

The characteristics of the primal and the dual problem can be stated as follows:

(i) The maximization problem in the primal becomes the minimization problem in the dual and *vice versa*.

(ii) The $\leq$ type constraints in the primal become $\geq$ type constraints in the dual and *vice versa*.

(iii) The co-efficients $c_1, c_2, \dots\dots, c_n$ in the objective function of the primal become the constants $b_1, b_2, \dots\dots, b_m$ in the constraints of the dual and *vice versa*.

(iv) If the primal has n variables and m constraints, then the dual has m variables and n constraints. Thus the body matrix of dual is the transpose of body matrix of primal and *vice versa*.

(v) A new set of variables appear in the dual.

(vi) The variables in both the primal and the dual are non-negative.

(vii) The dual of the dual problem is the original primal problem itself.

ILLUSTRATIVE EXAMPLES

Examples 1. *Obtain the dual of*

Maximize $Z = 5x_1 + 3x_2$

subject to

$$x_1 + x_2 \leq 2$$
$$5x_1 + 2x_2 \leq 10$$
$$3x_1 + 8x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

(M.D.U. Dec. 2011)

Sol. (i) Primal is of maximization type

Dual will be of 'minimization' type.

(ii) Primal has 2 variables x_1, x_2 and 3 constraints.

Dual will have 3 variables y_1, y_2, y_3 and 2 constraints.

(iii) Primal has $\leq$ constraints.

Dual will have $\geq$ constraints.

(iv) Primal has $c_1 = 5, c_2 = 3$ and $b_1 = 2, b_2 = 10, b_3 = 12$

Dual will have $c_1 = 2, c_2 = 10, c_3 = 12$ and $b_1 = 5, b_2 = 3$

The body matrix in the primal is $\begin{bmatrix} 1 & 1 \\ 5 & 2 \\ 3 & 8 \end{bmatrix}$

The body matrix in the dual will be

$$\begin{bmatrix} 1 & 1 \\ 5 & 2 \\ 3 & 8 \end{bmatrix}' = \begin{bmatrix} 1 & 5 & 3 \\ 1 & 2 & 8 \end{bmatrix}$$

Thus the dual is

Minimize $Z^* = 2y_1 + 10y_2 + 12y_3$

subject to

$$y_1 + 5y_2 + 3y_3 \geq 5$$
$$y_1 + 2y_2 + 8y_3 \geq 3$$

and $\quad y_1, y_2, y_3 \geq 0$

7.17. DUALITY PRINCIPLE

If either the primal or the dual problem has an optimal solution then so does the other problem and the optimal value of the primal's objective function is the same as that of its dual.

Example 2. *Solve the dual problem of the following LPP and hence find Max. Z:*

Maximize $Z = 20x_1 + 30x_2$

subject to

$$3x_1 + 3x_2 \leq 36$$
$$5x_1 + 2x_2 \leq 50$$
$$2x_1 + 6x_2 \leq 60$$
$$x_1, x_2 \geq 0.$$

Sol. The dual of the above LPP is:

Minimize $Z^* = 36y_1 + 50y_2 + 60y_3$

subject to

$$3y_1 + 5y_2 + 2y_3 \geq 20$$
$$3y_1 + 2y_2 + 5y_3 \geq 30$$
$$y_1, y_2, y_3 \geq 0$$

Solving this problem by simplex method, it can be verified that the final simplex table is:

		$c_j \rightarrow$	*36*	*50*	*60*	*0*	*0*	*M*	*M*
c_B	*Basis*	*Solution* $b\ (= y_B)$	y_1	y_2	y_3	s_1	s_2	A_1	A_2
36	y_1	5	1	$\frac{13}{6}$	0	$-\frac{1}{2}$	$\frac{1}{6}$	$\frac{1}{2}$	$-\frac{1}{6}$
60	y_3	$\frac{5}{2}$	0	$-\frac{3}{4}$	1	$\frac{1}{4}$	$-\frac{1}{4}$	$-\frac{1}{4}$	$\frac{1}{4}$
$Z^* = 330$		Z_j^*	36	33	60	-3	-9	3	9
		$C_j = c_j - Z_j^*$	0	17	0	3	9	M − 3	M − 9

The optimal solution to the dual problem is

$$y_1 = 5, y_2 = 0, y_3 = \frac{5}{2} \quad \text{and Min. } Z^* = 330$$

By duality principle = Max. Z = Min. Z* = 330.

7.18. DUAL SIMPLEX METHOD

Dual simplex method is similar to the regular simplex method which has already been discussed. The only difference lies in the criterion used for selecting the incoming and outgoing variables. In the dual simplex method, we first determine the outgoing variable and then the incoming variable whereas in the regular simplex method, the reverse is done. The regular simplex method starts with a basic feasible but non-optimal solution and works towards optimality, the dual simplex method starts with a basic infeasible but optimal solution and works towards feasibility.

7.19. WORKING PROCEDURE FOR DUAL SIMPLEX METHOD

Setp 1: Check whether the objective function is to be maximized or minimized. If the objective function is to be minimized, then convert it into maximization form.

Setp 2: Convert $\geq$ type constraints, if any, into $\leq$ type by multiplying such constraints by -1.

Step 3: Express all constraints as equations by adding slack variables, one for each constraint.

Step 4: Find the initial basic solution and express this information in the form of a table as in regular simplex method.

Step 5: Compute $C_j = c_j - Z_j$

(*a*) If all $C_j \leq 0$ and all $b_i \geq 0$, then the solution found above is the optimal basic feasible solution.

(*b*) If all $C_j \leq 0$ and at least one $b_i < 0$, then go to the next step.

(*c*) If any $C_j > 0$, the method fails.

Step 6: *Selection of key row and the outgoing variable*

Select the row containing the most negative b_i. This row is the key row and the basic variable heading the key row is the outgoing variable.

Step 7: *Selection of key column and the incoming variable*

(*a*) If all elements in the key row are ≥ 0, then the problem does not have feasible solution.

(*b*) If at least one element in the key row is negative, find the ratios of the corresponding elements of C_j-row to these elements. Ignore the ratios associated with positive or zero elements of the key row. Choose the smallest of these ratios. The corresponding column is the key column and the variable heading the key column is the incoming variable.

Step 8: Mark the key element and make it unity. Perform row operations as in the regular simplex method and repeat iterations until either an optimal feasible solution is attained or there is an indication of non-existence of a feasible solution.

ILLUSTRATIVE EXAMPLES

Example 1. *Using dual simplex method*

Maximize $Z = -3x_1 - x_2$

subject to $x_1 + x_2 \geq 1,\ 2x_1 + 3x_2 \geq 2,\ x_1, x_2 \geq 0.$

(M.D.U. Dec. 2008, Dec., 2009;K.U.K. Dec. 2009, Dec. 2010)

Sol.

Step 1. The problem is that of maximization.

Step 2. Converting $\geq$ type constraints into $\leq$ type, the L.P.P. takes the form

Maximize $Z = -3x_1 - x_2$

subject to $-x_1 - x_2 \leq -1,\ -2x_1 - 3x_2 \leq -2,\ x_1, x_2 \geq 0$

Step 3. Introducing slack variables s_1, s_2 (one for each constraint), the problem in standard form is:

Maximize $\qquad Z = -3x_1 - x_2 + 0s_1 + 0s_2$

subject to $\quad -x_1 - x_2 + s_1 = -1$

$$-2x_1 - 3x_2 + s_2 = -2$$

$$x_1, x_2, s_1, s_2 \geq 0$$

Step 4. The initial basic solution is given by

$$x_1 = x_2 = 0, S_1 = -1, S_2 = -2 \text{ at which } Z = 0$$

Dual Simplex Table I

C_B	Basis	$C_j \rightarrow$	-3	-1	0	0
		Solution $b (= x_B)$	x_1	x_2	s_1	s_2
0	S_1	-1	-1	-1	1	0
0	S_2	-2	-2	-3	0	1
$Z = 0$		Z_j	0	0	0	0
		$C_j = c_j - Z_j$	-3	-1	0	0

Step 5. Since all $C_j \leq 0$ and all $b_i < 0$, the initial solution is optimal but infeasible. So we proceed further.

Step 6. The most negative b_i (*i.e.*, the b_i which is negative and numerically largest) is $b_2 = -2$, **the second row is the key row and S_2 is the outgoing variable.**

Step 7. The key row has negative entries. The ratios of the elements in C_j-row to the corresponding negative elements of the key row (neglecting ratios corresponding to positive or zero elements of key row) are

$$\frac{-3}{-2} = \frac{3}{2},\ \frac{-1}{-3} = \frac{1}{3}$$

Smallest ratio is $\frac{1}{3}$. **The x_2-column is the key column and x_2 is the incoming variable.**

Step 8. The key element is -3, shown circled. Drop the outgoing variable s_2 from the basis and introduce the incoming variable x_2. The new basis contains s_1, x_2 as the basic variables. The co-efficient of x_2 in the objective function is -1. Therefore, the entry in C_B column corresponding to the new basic variable x_2 will be -1. Since the key element enclosed in the circle is not 1, divide all elements of the key row by the key element -3 to make it unity and make all other elements in the key column zero. Thus the key row

0	s_2	-2	-2	-3	0	1

is replaced by the new key row

-1	x_2	$\frac{2}{3}$	$\frac{2}{3}$	1	0	$-\frac{1}{3}$

Now we make all other elements of the key column (*i.e.*, x_2-column) zero.

Transformation of R_1. Key column entry in R_1 is -1

$\therefore$ $R_1\text{ (new)} = R_1\text{ (old)} - (-1)\, R_2\text{ (new)}$

$$-1 + 1\left(\frac{2}{3}\right) = -\frac{1}{3}$$

$$-1 + 1\left(\frac{2}{3}\right) = -\frac{1}{3}$$

$$-1 + 1\,(1) = 0$$

$$1 + 1\,(0) = 1$$

$$0 + 1\left(-\frac{1}{3}\right) = -\frac{1}{3}$$

The above information is summarized in the following table:

Dual Simplex Table II

C_B	Basis	c_j Solution $b\ (= x_B)$	-3 x_1	-1 x_2	0 s_1	0 s_2
0	s_1	$-\frac{1}{3}$	$-\frac{1}{3}$	0	1	$-\frac{1}{3}$
-1	x_2	$\frac{2}{3}$	$\frac{2}{3}$	1	0	$-\frac{1}{3}$
$Z = -\frac{2}{3}$		Z_j	$-\frac{2}{3}$	-1	0	$\frac{1}{3}$
		$C_j = c_j - Z_j$	$-\frac{7}{3}$	0	0	$-\frac{1}{3}$

Step 9. Test for optimality. Here all $C_j \le 0$ but $b_1 = -\frac{1}{3} < 0$, the current solution is not optimal. Therefore, an improvement in the value of Z is possible and we repeat steps 6 to 8.

The first row is the key row and s_1 is the outgoing variable. The ratios of the elements in C_j-row to the corresponding negative elements of the key row are

$$\frac{-\frac{7}{3}}{-\frac{1}{3}} = 7,\quad \frac{-\frac{1}{3}}{-\frac{1}{3}} = 1$$

The smallest ratio is 1, therefore, s_2-column is the key column and s_2 is the incoming variable. The key element is $-\frac{1}{3}$, shown circled. Drop s_1 and introduce s_2. The new basis contains s_2, x_2 as the basic variables. Dividing all elements of the key row by the key element $-\frac{1}{3}$, the new key row is

0	s_2	1	1	0	-3	1

Transformation of R_2. Key column entry in R_2 is $-\frac{1}{3}$.

$\therefore$ $R_2\text{ (new)} = R_2\text{ (old)} - \left(-\frac{1}{3}\right) R_1\text{ (new)}$

$$\frac{2}{3} + \frac{1}{3}(1) = 1$$

$$\frac{2}{3} + \frac{1}{3}(1) = 1$$

$$1 + \frac{1}{3}(0) = 1$$

$$0 + \frac{1}{3}(-3) = -1$$

$$-\frac{1}{3} + \frac{1}{3}(1) = 0$$

The above information is summarized in the following table:

Dual Simplex Table III

C_B	Basis	c_j Solution $b\ (= x_B)$	-3 x_1	-1 x_2	0 s_1	0 s_2
0	s_2	1	1	0	-3	1
-1	x_2	1	1	1	-1	0
$Z = -1$		Z_j	-1	-1	1	0
		$C_j = c_j - Z_j$	-2	0	-1	0

Here all $C_j \le 0$ and all $b_i > 0$, the optimal feasible solution has been attained. Thus the optimal feasible solution is

$$x_1 = 0,\ x_2 = 1,\ \text{Max. } Z = -1.$$

Example 2. *Using dual simplex method,*

Minimize $Z = x_1 + 2x_2 + 3x_3$

subject to $2x_1 - x_2 + x_3 \ge 4$

$x_1 + x_2 + 2x_3 \le 8$

$x_2 - x_3 \ge 2$

$x_1, x_2, x_3 \ge 0.$

Sol.

Step 1. The problem is that of minimization. Converting it into maximization problem by using the relationship

Min. $Z = -$ Max. Z^*, where $Z^* = -Z$, we have

$$\text{Maximize} \quad Z^* = -x_1 - 2x_2 - 3x_3$$

$$\text{subject to} \quad 2x_1 - x_2 + x_3 \geq 4$$
$$x_1 + x_2 + 2x_3 \leq 8$$
$$x_2 - x_3 \geq 2$$
$$x_1, x_2, x_3 \geq 0$$

Step 2. Converting $\geq$ type constraints into $\leq$ type, the L.P.P. takes the form

$$\text{Maximize } Z^* = -x_1 - 2x_2 - 3x_3$$
$$\text{subject to } -2x_1 + x_2 - x_3 \leq -4$$
$$x_1 + x_2 + 2x_3 \leq 8$$
$$-x_2 + x_3 \leq -2$$
$$x_1, x_2, x_3 \geq 0$$

Step 3. Introducing slack variables s_1, s_2, s_3, the problem in standard form is:

$$\text{Maximize } Z^* = -x_1 - 2x_2 - 3x_3 + 0s_1 + 0s_2 + 0s_3$$
$$\text{subject to} \quad -2x_1 + x_2 - x_3 + s_1 = -4$$
$$x_1 + x_2 + 2x_3 + s_2 = 8$$
$$-x_2 + x_3 + s_3 = -2$$
$$x_1, x_2, x_3, s_1, s_2, s_3 \geq 0$$

Step 4. The initial basic solution is given by

$$x_1 = x_2 = x_3 = 0,\ s_1 = -4,\ s_2 = 8,\ s_3 = -2$$

at which $Z^* = 0$

Dual Simplex Table I

C_B	Basis	$C_j \rightarrow$ Solution $b\ (= x_B)$	-1	-2	-3	0	0	0
			x_1	x_2	x_3	s_1	s_2	s_3
0	s_1	-4	-2	1	-1	1	0	0
0	s_2	8	1	1	2	0	1	0
0	s_3	-2	0	-1	1	0	0	1
$Z^* = 0$		Z_j^*	0	0	0	0	0	0
		$C_j = c_j - Z_j^*$	-1	-2	-3	0	0	0

Step 5. Since all $C_j \leq 0$ and at least one $b_i < 0$, the initial solution is optimal but infeasible. So we proceed further.

Step 6. The most negative b_i is $b_1 = -4$, the first row is the key row and s_1 is the outgoing variable.

Step 7. The key row has negative entries. The ratios of the elements in C_j-row to the corresponding negative elements of the key row are

$$\frac{-1}{-2} = \frac{1}{2},\ \frac{-3}{-1} = 3$$

Smallest ratio is $\frac{1}{2}$. The x_1-column is the key column and x_1 is the incoming variable.

Step 8. The key element is -2, shown circled. The new basis contains x_1, s_2, s_3 as the basic variables. Since the key element is not 1, divide all elements of the key row by the key element -2 to make it unity. Thus, the new key-row is

$$-1 \quad x_1 \quad 2 \quad 1 \quad -\frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2} \quad 0 \quad 0$$

Now we make all other elements of the key column zero.

Transformation of R_2. Key column entry in R_2 is 1.

$$\therefore \quad R_2\ (\text{new}) = R_2\ (\text{old}) - 1\ R_1\ (\text{new})$$

$$8 - 1\,(2) = 6,\quad 1 - 1\,(1) = 0,\quad 1 - 1\left(-\frac{1}{2}\right) = \frac{3}{2}$$

$$2 - 1\left(\frac{1}{2}\right) = \frac{3}{2},\quad 0 - 1\left(-\frac{1}{2}\right) = \frac{1}{2},\quad 1 - 1\,(0) = 1$$

$$0 - 1\,(0) = 0$$

Transformation of R_3. Key column entry in R_3 is 0.

$$\therefore \quad R_3\ (\text{new}) = R_3\ (\text{old})$$

The above information is summarized in the following table:

Dual Simplex Table II

C_B	Basis	$C_j \rightarrow$ Solution $b\ (= x_B)$	-1	-2	-3	0	0	0
			x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	2	1	$-\frac{1}{2}$	$\frac{1}{2}$	$-\frac{1}{2}$	0	0
0	s_2	6	0	$\frac{3}{2}$	$\frac{3}{2}$	$\frac{1}{2}$	1	0
0	s_3	-2	0	-1	1	0	0	1
$Z^* = -2$		Z_j^*	-1	$\frac{1}{2}$	$-\frac{1}{2}$	$\frac{1}{2}$	0	0
		$C_j = c_j - Z_j^*$	0	$-\frac{5}{2}$	$-\frac{5}{2}$	$-\frac{1}{2}$	0	0

Step 9. Here all $C_j \leq 0$ but $b_3 = -2 < 0$, the current solution is not optimal. So we repeat steps 6 to 8.

The third row is the key row and s_3 is the outgoing variable. The ratios of the elements in C_j-row to the corresponding negative elements of the key row are

$$\frac{-\frac{5}{2}}{-1} = \frac{5}{2}$$

The x_2-column is the key column and x_2 is the incoming variable. The key element is -1, shown circled. Drop s_3 and introduce x_2. The new basis contains x_1, s_2, x_2 as the basic variables. Dividing all elements of the key row by the key element -1, the new key row is

$-2 \quad x_2 \quad 2 \quad 0 \quad 1 \quad -1 \quad 0 \quad 0 \quad -1$

Now we make all other elements of the key column (*i.e.*, x_2-column) zero.

Transformation of R_1. Key column entry in R_1 is $-\frac{1}{2}$.

$$\therefore \quad R_1\,(\text{new}) = R_1\,(\text{old}) - \left(-\frac{1}{2}\right) R_3\,(\text{new})$$

$$2 + \frac{1}{2}(2) = 3, \quad 1 + \frac{1}{2}(0) = 1, \quad -\frac{1}{2} + \frac{1}{2}(1) = 0$$

$$\frac{1}{2} + \frac{1}{2}(-1) = 0, \quad -\frac{1}{2} + \frac{1}{2}(0) = -\frac{1}{2}, \quad 0 + \frac{1}{2}(0) = 0$$

$$0 + \frac{1}{2}(-1) = -\frac{1}{2}$$

Transformation of R_2. Key column entry in R_2 is $\frac{3}{2}$.

$$\therefore \quad R_2\,(\text{new}) = R_2\,(\text{old}) - \frac{3}{2} R_3\,(\text{new})$$

$$6 - \frac{3}{2}(2) = 3, \quad 0 - \frac{3}{2}(0) = 0, \quad \frac{3}{2} - \frac{3}{2}(1) = 0$$

$$\frac{3}{2} - \frac{3}{2}(-1) = 3, \quad \frac{1}{2} - \frac{3}{2}(0) = \frac{1}{2}, \quad 1 - \frac{3}{2}(0) = 1$$

$$0 - \frac{3}{2}(-1) = \frac{3}{2}$$

The above information is summarized in the following table:

Dual Simplex Table III

C_B	*Basis*	$C_j \rightarrow$	-1	-2	-3	0	0	0
		Solution $b\ (= x_B)$	x_1	x_2	x_3	s_1	s_2	s_3
-1	x_1	3	1	0	0	$-\frac{1}{2}$	0	$-\frac{1}{2}$
0	s_2	3	0	0	3	$\frac{1}{2}$	1	$\frac{3}{2}$
-2	x_2	2	0	1	-1	0	0	-1
$Z^* = -7$		Z_j^*	-1	-2	2	$\frac{1}{2}$	0	$\frac{5}{2}$
		$C_j = c_j - Z_j^*$	0	0	-5	$-\frac{1}{2}$	0	$-\frac{5}{2}$

Here all $C_j \leq 0$ and all $b_i > 0$, the optimal feasible solution has been attained. Thus the optimal feasible solution is

$$x_1 = 3,\ x_2 = 2,\ x_3 = 0,\ \text{Max. } Z^* = -7 \quad i.e., \text{ Min. } Z = 7.$$

EXERCISE 7.7

Using dual simplex method, solve the following problems:

1. Minimize $Z = 0.7x_1 + 0.5x_2$
   subject to $x_1 \geq 4,\ x_2 \geq 6,\ x_1 + 2x_2 \geq 20,\ 2x_1 + x_2 \geq 18,\ x_1, x_2 \geq 0$
2. Maximize $Z = -2x_1 - x_2$ (M.D.U. Dec. 2010)
   subject to $3x_1 + x_2 \geq 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \geq 3,\ x_1, x_2 \geq 0$
3. Maximize $Z = -3x_1 - 2x_2$
   subject to $x_1 + x_2 \geq 1,\ x_1 + x_2 \leq 7,\ x_1 + 2x_2 \geq 10,\ x_2 \leq 3,\ x_1, x_2 \geq 0$
4. Minimize $Z = 2x_1 + x_2$
   subject to $3x_1 + x_2 \geq 3,\ 4x_1 + 3x_2 \geq 6,\ x_1 + 2x_2 \leq 3,\ x_1, x_2 \geq 0$
5. Minimize $Z = 2x_1 + 2x_2 + 4x_3$
   subject to $2x_1 + 3x_2 + 5x_3 \geq 2,\ 3x_1 + x_2 + 7x_3 \leq 3,\ x_1 + 4x_2 + 6x_3 \leq 5,\ x_1, x_2, x_3 \geq 0.$
6. Minimize $Z = 10x_1 + 6x_2 + 2x_3$ (M.D.U. May 2011)
   subject to $-x_1 + x_2 + x_3 \geq 1,\ 3x_1 + x_2 - x_3 \geq 2,\ x_1, x_2, x_3 \geq 0.$

Answers

1. $x_1 = \frac{16}{3},\ x_2 = \frac{22}{3}$; min. $Z = 7.4$
2. $x_1 = \frac{3}{5},\ x_2 = \frac{6}{5}$, Max. $Z = -\frac{12}{5}$
3. $x_1 = 4,\ x_2 = 3$, Max. $Z = -18$
4. $x_1 = \frac{3}{5},\ x_2 = \frac{6}{5}$, Min. $Z = \frac{12}{5}$
5. $x_1 = 0,\ x_2 = \frac{2}{3},\ x_3 = 0$, Min. $Z = \frac{4}{3}$
6. $x_1 = \frac{1}{4},\ x_2 = \frac{5}{4},\ x_3 = 0$, Min. $Z = \frac{4}{3}$